# VL: GEOMETRIE UND ANALYSIS VON SUPERMANNIGFALTIGKEITEN UND LIE-SUPERGRUPPEN 

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#### Abstract

Die Vorlesung folgt dem Manuskript eines Buchprojekts mit J. Hilgert (Paderborn) und T. Wurzbacher (Bochum/Metz).

Das Skriptum ist nicht korrekturgelesen und nur zum internen Gebrauch bestimmt.


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## 1. Introduction

## 2. Categories

### 2.1. Categories, functors, and natural transformations.

Definition 2.1. A category $\mathcal{C}$ consists of the following data:

- A class $\operatorname{ObC}$, the elements of which are called objects;
- for any pair $(X, Y)$ of objects, a set $\operatorname{Hom}_{\mathcal{C}}(X, Y)=\operatorname{Hom}(X, Y)$, the elements of which are denoted $f: X \rightarrow Y$ and called morphisms from $X$ to $Y$;
- and for any triple $(X, Y, Z)$ of objects, a composition map

$$
\circ: \operatorname{Hom}(Y, Z) \times \operatorname{Hom}(X, Y) \rightarrow \operatorname{Hom}(X, Z):(g, f) \mapsto g \circ f
$$

-fulfilling the following conditions:
(i). For any object $X$, there is a morphism $\operatorname{id}_{X}=\mathrm{id}: X \rightarrow X$, called the identity of $X$, such that $f \circ \mathrm{id}=f$ and id $\circ g=g$ whenever this makes sense;
(ii). composition is associative, that is, $(h \circ g) \circ f=h \circ(g \circ f)$ whenever this makes sense.

It is a standard fact that the morphisms $\mathrm{id}_{X}$ are unique. A morphism $f: X \rightarrow Y$ is called an isomorphism if there exists a morphism $g: Y \rightarrow X$ such that $g \circ f=\operatorname{id}_{X}$ and $f \circ g=\mathrm{id}_{Y}$. In this case, $g$ is unique, and we denote $g=f^{-1}$. One writes $X \cong Y$ if there exists an isomophism $X \rightarrow Y$; in this case, one says that $X$ and $Y$ are isomorphic.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is given by:

- a subclass $\operatorname{Ob} \mathcal{D} \subset \mathcal{C}$ and
- subsets $\operatorname{Hom}_{\mathcal{D}}(X, Y) \subset \operatorname{Hom}_{\mathcal{C}}(X, Y)$, for any $X, Y \in \operatorname{Ob} \mathcal{D}$
-such that:
(i). For any object $X \in \operatorname{Ob} \mathcal{D}, \operatorname{id}_{X} \in \operatorname{Hom}_{\mathcal{D}}(X, X)$,
(ii). and $g \circ f \in \operatorname{Hom}_{\mathcal{D}}(Z, X)$ for any $g \in \operatorname{Hom}_{\mathcal{D}}(Z, Y), f \in \operatorname{Hom}_{\mathcal{D}}(Y, X)$.

Clearly, any subcategory of a category is itself a category. A subcategory $\mathcal{D}$ of a category $\mathcal{C}$ is called full if $\operatorname{Hom}_{\mathcal{D}}(X, Y)=\operatorname{Hom}_{\mathcal{C}}(X, Y)$ for all objects $X, Y$ of $\mathcal{D}$. So a full subcategory is specified just by a subclass of the objects.

## Example 2.2.

(1). The category Sets is the category of sets and maps, with the usual composition. A full subcategory is given by the finite sets.

Other examples of subcategories of Sets are vector spaces and linear maps, groups and group homomorphisms, rings and ring homomorphisms, topological spaces and continuous maps, etc. None of these is full.

An example of a full subcategory of the category of vector spaces is the category of finite-dimensional vector spaces.
(2). Any group $G$ gives rise to a category $\mathcal{C}$, where we take $\operatorname{Ob} \mathcal{C}=\{1\}$, $\operatorname{Hom}(1,1)=G$, and the composition to be the group law. In this category, any morphism is an isomorphism. (Such a category is called a groupoid.) The category $\mathcal{C}$ is an example of a category which is not a subcategory of Sets (although, in a sense to be defined, it is isomorphic to such a subcategory).
(3). Let $X$ be a set. A preorder $\leqslant$ on $X$ is a reflexive and transitive relation. It is called an order (resp. equivalence relation) if in addition, it is antisymmetric (resp. symmetric).

One defines a category $\mathcal{C}$ by taking $\operatorname{Ob} \mathcal{C}=X$ and

$$
\operatorname{Hom}(x, y)= \begin{cases}\{(x, y)\} & x \leqslant y \\ \varnothing & \text { otherwise }\end{cases}
$$

This already determines the composition on $\mathcal{C}$. (It is given by the rule $(z, y) \circ(y, x)=(z, x)$ for composable morphisms.)

If $X$ is arbitrary, there are some universally existant preorders. For instance, $\leqslant$ might just be equality. (As a subset of $X \times X$, this is the diagonal.) At the other extreme, one may take $x \leqslant y$ to be verified for all $x, y \in X$. (As a subset of $X \times X$, this is all of $X \times X$.) These are examples of equivalence relations on $X$; in fact, any equivalence relation is a preorder.

In fact, it is easy to see that $\mathcal{C}$ is a groupoid if and only if $\leqslant$ is an equivalence relation. At the other extreme, $\leqslant$ is an order if and only if the only isomorphisms in $\mathcal{C}$ are $\mathrm{id}_{x}, x \in X$.

There is an amusing equivalence relation on $X \times X$. Namely, define $\leqslant$ by

$$
(x, y) \leqslant(u, v) \quad: \Leftrightarrow \quad(x, y)=(u, v) \text { or }(x, y)=(v, u)
$$

We will have occasion to use it in the proof of Proposition 3.18.
Exercise 2.3. Let $\mathcal{C}$ be a category such that $X=\mathrm{Ob} \mathcal{C}$ is a set. Show that if $\mathcal{C}$ is discrete, i.e. $\operatorname{Hom}(x, y)$ has at most one element for all $x, y \in X$, then (up to some identification of hom-sets) $\mathcal{C}$ is associated with a preorder.

Definition 2.4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is given by the following data:

- A map $F: \mathrm{ObC} \rightarrow \mathrm{Ob} \mathcal{D}$ and
- maps $\operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$, also denoted by $F$ -such that:
(i). For any $X \in \operatorname{Ob} \mathcal{C}, F\left(\mathrm{id}_{X}\right)=\mathrm{id}_{F(X)}$, and
(ii). for any morphisms $f: X \rightarrow Y, g: Y \rightarrow Z$ in $\mathcal{C}$, one has the equality $F(g \circ f)=F(g) \circ F(f)$.
One also says that $F$ is a covariant functor. A contravariant functor is a functor $\mathcal{C}^{o p} \rightarrow \mathcal{D}$. By definition, any (covariant or contravariant) functor sends isomorphisms to isomorphisms.

For any category $\mathcal{C}$, there is an identity functor $\mathrm{id}_{\mathcal{C}}=\mathrm{id}: \mathcal{C} \rightarrow \mathcal{C}$ which is the identity on objects and morphisms. There is also a contravariant functor $\mathcal{C} \rightarrow \mathcal{C}^{o p}$, given as the identity functor of $\mathcal{C}^{o p}$. A functor is called an isomorphism of categories if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that $G \circ F=\mathrm{id}_{\mathcal{C}}$ and $F \circ G=\mathrm{id}_{\mathcal{D}}$.

Isomorphisms of categories are rather rare, and it is usually more convenient to consider a weaker form of equivalence, defined as follows.

A functor is called fully faithful if for any $X, Y \in \mathrm{Ob} \mathcal{C}$, the map on morphisms $F: \operatorname{Hom}_{\mathcal{C}}(X, Y) \rightarrow \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$ is bijective. It is called essentially surjective if for any object $Y$ of $\mathcal{D}$, there exists $X \in \operatorname{Ob} \mathcal{C}$ such that $F(X)$ is isomorphic to $Y$. It is called an equivalence of categories if and only if it is fully faithful and essentially surjective.

## Example 2.5.

(1). The embedding of a full subcategory is a fully faithful functor.
(2). Let $k$ be a field. Let $\mathcal{C}$ be the category whose objects are $k^{n}, n \in \mathbb{N}$, and whose hom-sets are $\operatorname{Hom}_{\mathcal{C}}\left(k^{n}, k^{m}\right)=k^{m \times n}$, where composition is given by matrix multiplication. Let $\mathcal{D}$ be the category of finite-dimensional $k$-vector spaces and linear maps. Define $F\left(k^{n}\right)=k^{n}$ and $F(A), A \in k^{m \times n}$, to be the linear map $k^{n} \rightarrow k^{m}, x \mapsto A x$ given by multiplication of the matrix $A$ with column vectors. Then $F$ is an equivalence of categories.
(3). The natural contravariant functor $\mathcal{C} \rightarrow \mathcal{C}^{o p}$ is a contravariant isomorphism. (It is an isomorphism in the category opposite to the category of categories, see below.)
(4). For categories associated with groups $G, H$, a functor $F$ is the same as a group homomorphism. In this special case, any functor is bijective on objects and essentially surjective. Hence, the following are equivalent: $F$ is a group isomorphism, $F$ is an isomorphism of categories, $F$ is an equivalence of categories, and $F$ is fully faithful.
(5). Given any category $\mathcal{C}$, one defines the opposite category $\mathcal{C}^{o p}$ by taking $\operatorname{Ob} \mathcal{C}^{o p}=\operatorname{ObC}, \operatorname{Hom}_{\mathcal{C}^{o p}}(X, Y)=\operatorname{Hom}_{\mathcal{C}}(Y, X)$, and $g \circ^{o p} f=f \circ g$ as the composition. Clearly, $\left(\mathcal{C}^{o p}\right)^{o p}=\mathcal{C}$.

If $\mathcal{C}$ is the category associated with a group $G$, then $\mathcal{C}^{o p}$ is associated with the opposite group. One has $\mathcal{C}=\mathcal{C}^{o p}$ if and only $G$ is Abelian.

If $\mathcal{C}$ is the category associated with a preorder, then $\mathcal{C}^{o p}$ is associated with the opposite relation. One has $\mathcal{C}=\mathcal{C}^{o p}$ if and only if $\leqslant$ is an equivalence relation.
(6). If $X \in \operatorname{Ob\mathcal {C}}$, then $\operatorname{Hom}(X,-)$ is a covariant functor $\mathcal{C} \rightarrow$ Sets, and $\operatorname{Hom}(-, X)$ is a contravariant functor $\mathcal{C} \rightarrow$ Sets. On morphisms $f: Y \rightarrow Z$,

$$
\begin{aligned}
& \operatorname{Hom}(X, f): \operatorname{Hom}(X, Y) \rightarrow \operatorname{Hom}(X, Z): g \mapsto f \circ g \\
& \operatorname{Hom}(f, X): \operatorname{Hom}(Z, X) \rightarrow \operatorname{Hom}(Y, X): g \mapsto g \circ f
\end{aligned}
$$

We will use the functor $\operatorname{Hom}(-, X)$ quite frequently, so it is useful to introduce a less cumbersome and more intuitive notation: Namely, we write $X(-)=\operatorname{Hom}(-, X)$. For $Y \in \operatorname{Ob} \mathcal{C}, X(Y)$ is called the set of $Y$-points of $X$. We write $y \in_{Y} X$ for the relation $y \in X(Y)$.

This makes sense for the following reason: For $\mathcal{C}=$ Sets, let $*$ the singleton set. (Of course, there many such sets, but they are all isomorphic.) An point $x \in X$ is the same as a map $* \rightarrow X$. In other words, $X(*)=X$, so that the 'generalised points' (that is, the elements of $X(Y)$ for some $Y$ ), are really a generalisation of the ordinary points of $X$.

As we shall see below, more is true: The concept of generalised points will allow us to consider any object in any category as a (parameter-dependent) set, and any morphism as a (parameter-dependent) map of sets.

Definition 2.6. Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors (contravariant functors). A natural transformation or functor morphism $\theta: F_{1} \rightarrow F_{2}$ is given by morphisms $\theta_{X}: F(X) \rightarrow G(X)$ in $\mathcal{D}$, for each $X \in \operatorname{Ob\mathcal {C}}$, subject to the following condition: For any morphism $f: X \rightarrow Y$ in $\mathcal{C}$, the following
diagram is commutative:

(In the contravariant case, the orientation of the vertical arrows is inverted.)
Given natural transformations $\theta: F \rightarrow G, \theta^{\prime}: G \rightarrow H$, one defines the horizontal composition $\theta^{\prime} \circ \theta: F \rightarrow H$ by $\left(\theta^{\prime} \circ \theta\right)_{X}=\theta_{X}^{\prime} \circ \theta_{X}$.

Setting aside set-theoretical objections, there is a category of categories and natural transformations. A natural transformation is called a natural equivalence or functor isomorphism if it is an isomorphism in this category.
Exercise 2.7. A functor $G: \mathcal{D} \rightarrow \mathcal{C}$ is called a quasi-inverse of $F: \mathcal{C} \rightarrow \mathcal{D}$ if $G \circ F \cong \operatorname{id}_{\mathcal{C}}$ and $F \circ G \cong \operatorname{id}_{\mathcal{D}}$. Show that $F$ is an equivalence of categories if and only if it possesses a quasi-inverse.
Definition 2.8. A functor (resp. contravariant functor) $F: \mathcal{C} \rightarrow$ Sets is called representable if there is $X \in \operatorname{Ob} \mathcal{C}$ such that $F \cong \operatorname{Hom}(X,-)$ (resp. $F \cong$ $\operatorname{Hom}(-, X))$. In this case, $X$ is unique up to canonical isomorphism, and is called a representative of $F$.
2.9. For a category $\mathcal{C}$, let $\mathcal{C}^{\vee}$ denote the category of contravariant functors $\mathcal{C} \rightarrow$ Sets and natural transformations. Define a functor $h: \mathcal{C} \rightarrow \mathcal{C}^{\vee}$ by $h(X)=\operatorname{Hom}(-, X)=X(-)$ and

$$
\begin{gathered}
h(f: X \rightarrow Y)=\operatorname{Hom}(-, f): \operatorname{Hom}(-, X) \rightarrow \operatorname{Hom}(-, Y), \\
h(f)_{Z}=\operatorname{Hom}(Z, f): \operatorname{Hom}(Z, X) \rightarrow \operatorname{Hom}(Z, Y) .
\end{gathered}
$$

While the above notation is more or less standard, we will want to introduce a more succinct terminology, in keep with our policy concerning generalised points: For a morphism $f: X \rightarrow Y$ and $z \in_{Z} X$, we denote $f(z)=f \circ z$. Then $h(f)_{Z}=\operatorname{Hom}(Z, f)$ is simply the maps of sets $f(-)$ which sends $z \in_{Z} X$ to $f(z) \in_{Z} Y$. Moreover, the functoriality of $h: X \mapsto X(-)$, $f \mapsto f(-)$ is expressed by the equation $(g \circ f)(z)=g(f(z))$.

The following proposition is known as the Yoneda Lemma.
Proposition 2.10. For any $X \in \mathrm{Ob} \mathrm{\mathcal{C}}$ and $F \in \mathrm{Ob}^{\vee}$, we have

$$
\operatorname{Hom}_{\mathcal{C}}{ }^{\vee}(X(-), F) \cong F(X) .
$$

In particular, $h$ is fully faithful.
Proof. Let us define a map $\phi: \operatorname{Hom}_{\mathcal{C}}(X(-), F) \rightarrow F(X)$. To that end, we observe that for $\eta \in \operatorname{Hom}_{\mathcal{C}} \vee(X(-), F), \eta_{X}: X(X) \rightarrow F(X)$. So all depends on the choice of an $X$-point of $X$. The only canonical such choice the so-called generic point $x \in_{X} X$, which is $x=\operatorname{id}_{X}$. So we set $\phi\left(\eta_{X}\right)=\eta_{X}(x)$.

Conversely, for $y \in F(X)$, we let $\psi(y)$ be the natural transformation consisting of the maps $\psi(y)_{Z}: X(Z) \rightarrow F(Z)$ which send $z: Z \rightarrow X$ to $F(z)(y)$. Since $F(x)=\operatorname{id}_{F(X)}$, we have that $\phi(\psi(y))=\psi(y)_{X}(x)=y$. Finally, if $y=\eta_{X}(x)$, then

$$
\psi(y)_{Z}(z)=F(z)(y)=F(z)\left(\eta_{X}(x)\right)=\eta_{Z}(z(x))=\eta_{Z}(z)
$$

by the naturality of $\eta$. This proves that $\phi$ and $\psi$ are mutually inverse, and establishes the asserted bijection.

To see that $h$ is fully faithful, we insert $Y(-)$ for $F$ to obtain

$$
\operatorname{Hom}_{\mathcal{C}^{\vee}}(X(-), Y(-)) \cong Y(X)=\operatorname{Hom}_{\mathcal{C}}(X, Y)
$$

We need to check that the inverse bijection (viz. $\psi$ ) is just the hom-set map of $h$. So, for $f: X \rightarrow Y$, compute

$$
\psi(f)_{Z}(z)=Y(z)(f)=f(z)=h(f)_{Z}(z)
$$

This proves the claim.
Remark 2.11.
(1). The contravariant functors $F$ in the essential image of $h$, i.e. $F \cong X(-)$ for some $X \in \mathrm{Ob} \mathcal{C}$, are exactly the representable ones.
(2). An important use of the Yoneda Lemma is in defining morphisms. A morphism $f: X \rightarrow Y$ in any category is determined uniquely by its values $f(s) \epsilon_{S} Y$ on generalised points $s \in_{S} X$. (A tautology: One has $f(x)=f$, for $x \in_{X} X$ the generic point.) But more is true: Given set maps $f_{S}: X(S) \rightarrow Y(S), s \mapsto f_{S}(s)$, there is a (unique) morphism $f: X \rightarrow Y$ such that $f(s)=f_{S}(s)$ for all $s \in_{S} X$, if and only if the collection $f_{S}$ is natural, that is, if for any morphism $g: T \rightarrow S, f_{T}(s \circ g)=f_{S}(s) \circ g$.

### 2.2. Adjoints and limits.

Exercise 2.12. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ and $G: \mathcal{D} \rightarrow \mathcal{C}$ be functors. Prove that the following are equivalent:
(1). There are natural transformations

$$
\alpha: F \circ G \rightarrow \operatorname{id}_{\mathcal{D}} \quad \text { and } \quad \beta: \mathrm{id}_{\mathcal{C}} \rightarrow G \circ F
$$

such that the following composites are the identity transformation:

$$
G \xrightarrow{\beta G} G F G \xrightarrow{G \alpha} G
$$

and

$$
F \xrightarrow{F \beta} F G F \xrightarrow{\alpha F} F
$$

Here, $(\beta G)_{Y}=\beta_{G(Y)},(G \alpha)_{X}=G\left(\alpha_{X}\right)$, etc.
(2). There is a natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(F-,-) \cong \operatorname{Hom}_{\mathcal{C}}(-, G-)$ as functors $\mathcal{C}^{o p} \times \mathcal{D} \rightarrow$ Sets.

We will frequently express the latter statement as follows: "There is a natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(F(X), Y) \cong \operatorname{Hom}_{\mathcal{C}}(X, G(Y))$, as $X$ runs through $\mathcal{C}$ and $Y$ runs through $\mathcal{D}$."

Definition 2.13. In the situation of the above exercise, $F$ is called a left adjoint of $G$, and $G$ is called a right adjoint of $F$. The pair $(F, G)$ is called an adjunction. The natural transformations $\alpha$ and $\beta$ are called the counit and unit, respectively, of the adjunction.

Proposition 2.14. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then $F$ possesses a right adjoint $G$ (say) if and only for any $Y \in \operatorname{Ob} \mathcal{D}$, the contravariant functor $\operatorname{Hom}_{\mathcal{D}}(F-, Y)$ is representable. In this case, $G(Y)$ is a representative.
$A$ similar statement holds for the existence of left adjoints of $G$.

Remark 2.15. As the proposition shows, adjoints are determined uniquely up to natural equivalence.

Proof of Proposition 2.14. The condition is clearly necessary, by part (2) of Exercise 2.12. Conversely, let $G(B)$ be a representative of $\operatorname{Hom}(F-, B)$. Then $\theta_{X}^{B}: \operatorname{Hom}_{\mathcal{C}}(X, G(B)) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{D}}(F(X), B)$. In particular,

$$
\begin{aligned}
& \theta_{G(A)}^{B}: \operatorname{Hom}_{\mathcal{C}}(G(A), G(B)) \stackrel{\cong}{\cong} \operatorname{Hom}_{\mathcal{D}}(F(G(A)), B), \\
& \theta_{X}^{F(Y)}: \operatorname{Hom}_{\mathcal{C}}(X, G(F(Y))) \stackrel{\cong}{\cong} \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y)) .
\end{aligned}
$$

Since $\theta^{B}$ is a natural transformation for any $B$, we have for $h: Z \rightarrow X$,

$$
\operatorname{Hom}_{\mathcal{D}}(F(h), B) \circ \theta_{X}^{B}=\theta_{Z}^{B} \circ \operatorname{Hom}_{\mathcal{C}}(h, G(B))
$$

This amounts, for $f: X \rightarrow G(B)$ and $h: Z \rightarrow X$, to the equation

$$
\begin{equation*}
\theta_{X}^{B}(f) \circ F(h)=\theta_{Z}^{B}(f \circ h) \tag{2.1}
\end{equation*}
$$

We define $\alpha_{A}: F(G(A)) \rightarrow A$ and $\beta_{X}: X \rightarrow G(F(X))$ by

$$
\alpha_{A}=\theta_{G(A)}^{A}\left(\operatorname{id}_{G(A)}\right) \quad \text { and } \quad \theta_{X}^{F(X)}\left(\beta_{X}\right)=\operatorname{id}_{F(X)}
$$

(In passing, observe that $\alpha_{A}$ corresponds to $\theta^{A}$ under Yoneda's isomorphism.) Furthermore, for $g: A \rightarrow B$, define $G(g): G(A) \rightarrow G(B)$ by

$$
\theta_{G(A)}^{B}(G(g))=g \circ \alpha_{A}
$$

For $g: A \rightarrow B$, we have by Equation 2.1,

$$
\begin{aligned}
\alpha_{B} \circ F(G(g)) & =\theta_{G(B)}^{B}\left(\operatorname{id}_{G(B)}\right) \circ F(G(g)) \\
& =\theta_{G(A)}^{B}\left(\operatorname{id}_{G(B)} \circ G(g)\right)=\theta_{G(A)}^{B}(G(g))=g \circ \alpha_{A}
\end{aligned}
$$

This and Equation 2.1 allow us to compute, for $g: A \rightarrow B$ and $h: B \rightarrow C$,

$$
\begin{aligned}
\theta_{G(A)}^{C}(G(h) \circ G(g)) & =\theta_{G(B)}^{C}(G(h)) \circ F(G(g))=h \circ\left(\alpha_{B} \circ F(G(g))\right) \\
& =(h \circ g) \circ \alpha_{A}=\theta_{G(A)}^{C}(G(h \circ g)) .
\end{aligned}
$$

Since $\theta_{G(A)}^{A}\left(G\left(\operatorname{id}_{A}\right)\right)=\operatorname{id}_{A} \circ \alpha_{A}=\alpha_{A}=\theta_{G(A)}^{A}\left(\operatorname{id}_{G(A)}\right)$ by definition, this shows that $G$ is a functor.

Now, we have already proved that $\alpha: F G \rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural transformation. We have, again by Equation 2.1,

$$
\begin{aligned}
\alpha_{F(X)} \circ F\left(\beta_{X}\right) & =\theta_{G(F(X))}^{F(X)}\left(\mathrm{id}_{G(F(X))}\right) \circ F\left(\beta_{X}\right) \\
& =\theta_{X}^{F(X)}\left(\operatorname{id}_{G(F(X))} \circ \beta_{X}\right)=\theta_{X}^{F(X)}\left(\beta_{X}\right)=\operatorname{id}_{F(X)}
\end{aligned}
$$

This, together with Equation 2.1, allows us to compute, for $f: X \rightarrow Y$,

$$
\begin{aligned}
\theta_{X}^{F(Y)}\left(G(F(f)) \circ \beta_{X}\right) & =\theta_{G(F(X))}^{F(Y)}(G(F(f))) \circ F\left(\beta_{X}\right) \\
& =F(f) \circ \alpha_{F(X)} \circ F\left(\beta_{X}\right)=\operatorname{id}_{F(Y)} \circ F(f) \\
& =\theta_{Y}^{F(Y)}\left(\beta_{Y}\right) \circ F(f)=\theta_{X}^{F(Y)}\left(\beta_{Y} \circ f\right)
\end{aligned}
$$

so that $\beta: \operatorname{id}_{\mathcal{C}} \rightarrow G F$ is a natural transformation.

We have already proved that $\alpha F \circ F \beta=\mathrm{id}_{F}$. Using this relation and once again Equation 2.1, we compute,

$$
\begin{aligned}
\theta_{G(A)}^{A}\left(G\left(\alpha_{A}\right) \circ \beta_{G(A)}\right) & =\theta_{G(F(G(A)))}^{A}\left(G\left(\alpha_{A}\right)\right) \circ F\left(\beta_{G(A)}\right) \\
& =\alpha_{A} \circ \alpha_{F(G(A))} \circ F\left(\beta_{G(A)}\right)=\alpha_{A}=\theta_{G(A)}^{A}\left(\operatorname{id}_{G(A)}\right)
\end{aligned}
$$

which proves that $G \alpha \circ \beta G=\operatorname{id}_{G}$. We have checked all the conditions in part (1) of Exercise 2.12, and this proves the assertion.

Example 2.16. Let $\mathcal{D}$ be the category of left modules and module homomorphisms over some ring $R$. Let $G: \mathcal{D} \rightarrow$ Sets which assigns to each module the underlying set and to each module homomorphism the underlying map of sets. (This is an example of which is called a forgetful functor.)

Then $G$ has a left adjoint, namely the functor $F$ which assigns to each set $X$ the free left $R$-module $F(X)=R\langle X\rangle$, and to each maps $f: X \rightarrow Y$ the unique $R$-module homomorphism $F(f): R\langle X\rangle \rightarrow R\langle Y\rangle$ such that $F(f)(x)=f(x)$ for all $x \in X$.

In particular, for $R=\mathbb{Z}, \mathcal{D}$ is the category of Abelian groups, and $F$ gives the free Abelian group of a set.

Remark 2.17. The preceding example indicates that adjoint functors provide a systematic way of dealing with universal constructions.

Namely, if $(F, G)$ is an adjunction, then for each $X \in \operatorname{Ob\mathcal {C}}$, we have $F(X) \in \mathcal{D}$ and a morphism $\beta_{X}: X \rightarrow G(F(X))$ satisfying the following universal property: For any $A \in \operatorname{Ob} \mathcal{D}$ and any morphism $f: X \rightarrow G(A)$, there is a unique morphism $\tilde{f}: A \rightarrow F(X)$ such that $f=G(\tilde{f}) \circ \beta_{X}$. That is, the following diagram commutes:


The dual statement holds for $G(A)$ and $\alpha_{A}$. Both can be derived from the proof of Proposition 2.14.

Definition 2.18. Let $(J, \leqslant)$ be a preordered set, and $\mathcal{C}$ a category. A projective system $\left(X_{i}, f_{i j}\right)$ in $\mathcal{C}$, indexed over $J$, consists of $X_{i} \in \mathrm{Ob} \mathcal{C}, i \in J$, and of morphisms $f_{i j}: X_{j} \rightarrow X_{i}$ for $i \leqslant j$, such that

$$
f_{i i}=\mathrm{id} \quad \text { and } \quad f_{i j} \circ f_{j k}=f_{i k} \quad \text { for all } i \leqslant j \leqslant k
$$

Thus, if we denote by $J$ the category associated with the preorder, a projective system is just a contravariant functor $J \rightarrow \mathcal{C}$. Similarly, a functor $J \rightarrow \mathcal{C}$ is called an inductive system; that is, it consists of objects $X_{i}, i \in J$, and morphisms $f_{i j}: X_{i} \rightarrow X_{j}$ for $i \leqslant j$, such that $f_{i i}=$ id and $f_{j k} \circ f_{i j}=f_{i k}$.

Let $\left(X_{i}, f_{i j}\right)$ be a projective system in $\mathcal{C}$, indexed by $J$. A projective limit of this system is an object $X \in \mathrm{Ob} \mathcal{C}$, together with morphisms $f_{i}: X \rightarrow X_{i}$, $i \in J$, such that $f_{i}=f_{i j} \circ f_{j}$, satisfying the following universal property: For any $Y \in \mathrm{ObC}$ and morphisms $g_{i}: Y \rightarrow X_{i}, i \in J$, such that $g_{i}=f_{i j} \circ g_{j}$, there is a unique morphism $g: Y \rightarrow X$, making the following diagram
commutative:


Dually, an inductive limit of an inductive system $\left(X_{i}, f_{i j}\right)$ in $\mathcal{C}$, indexed by $J$, is given by an object $X$ and morphisms $f_{i}: X_{i} \rightarrow X$, satisfying $f_{i}=f_{j} \circ f_{i j}$ and the universal property encoded by the following commutative diagram:

2.19. Let $\mathcal{C}$ be any category, and $J$ be a category whose object class is a set. We let $\mathcal{C}^{J}$ be the category of functors $J \rightarrow \mathcal{C}$ and their natural transformations.

We define the diagonal functor $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{J}$ as follows: For $X \in \mathrm{Ob} \mathcal{C}$, $\Delta(X)$ is the constant functor $j \mapsto X,(f: i \rightarrow j) \mapsto \mathrm{id}_{X}$; for $h: X \rightarrow Y$, $\Delta(h)$ is the natural transformation given by $\Delta(h)_{j}=h$ for all $j \in \operatorname{Ob} J$.

Proposition 2.20. Let $(J, \leqslant)$ be a preordered set and $\mathcal{C}$ a category.
(1). Let $F: J^{o p} \rightarrow \mathcal{C}$ be a projective system. Then $F$ has a projective limit if and only if the functor $\operatorname{Hom}_{\mathcal{C}^{J^{o p}}}(\Delta-, F)$ is representable. In this case, the projective limit is a representative, and in particular, unique up to canonical isomorphism.
(2). Let $F: J \rightarrow \mathcal{C}$ be an inductive system. Then $F$ has an inductive limit if and only if the contravariant functor $\operatorname{Hom}_{\mathcal{C}^{J}}(F, \Delta-)$ is representable. In this case, the inductive limits is a representative, and in particular, unique up to canonical isomorphism.

Proof. We only prove (1), since (2) is the dual statement (take $\mathcal{C}$ to $\mathcal{C}^{o p}$ ).
First, observe that the elements of $\operatorname{Hom}_{\mathcal{C}^{J o p}}(\Delta(Y), F)$ are simply families of commutative diagrams for $i \leqslant j$,

which may also be written in triangular form.
Now, we may use the same construction as in the proof of Proposition 2.14. If we have a representative $X$, then by definition, there is a natural equivalence $\theta: \operatorname{Hom}_{\mathcal{C}^{J^{o p}}}(\Delta-, F) \rightarrow X(-)$. In particular, there is a natural transformation $f: \Delta(X) \rightarrow F$, given by $\theta_{X}(f)=\mathrm{id}_{X}$; this gives a collection
of morphisms $f_{i}: X \rightarrow X_{i}$ such that $f_{i}=f_{i j} \circ f_{j}$ whenever $i \leqslant j$. Moreover, the $\left(g_{j}\right)$ as the above diagram correspond via $\theta_{Y}$ bijectively to $g: Y \rightarrow X$ such that $f_{i} \circ g=g_{i}$. This proves that $X, f_{i}$ are a projective limit of $F$.

Conversely, if $X$ is a projective limit, then the above discussion gives a recipe for the construction of bijections $\theta_{Y}: \operatorname{Hom}_{\mathcal{C}^{J o p}}(\Delta(Y), F) \rightarrow X(Y)$, and it is straightforward to check that they form a natural transformation. This completes the proof.
Definition 2.21. If ( $X_{i}, f_{i j}$ ) is a projective (resp. an inductive) system, then its limit, if it exists, is denoted by $\lim _{\underset{j}{ }} X_{j}$ (resp. $\lim _{j} X_{j}$ ).
Corollary 2.22. Let $\mathcal{C}$ be a category and $J$ a preodered set. Then any projective (resp. inductive) system in $\mathcal{C}$ indexed over $J$ possesses a projective (resp. an inductive) limit if and only if the diagonal functor $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{J^{o p}}$ (resp. $\Delta: \mathcal{C} \rightarrow \mathcal{C}^{J}$ ) possesses a right adjoint $G$ (resp. a left adjoint $F$ ).

In this case, we have

$$
\lim _{\rightleftarrows} X_{j}=G\left(\left(X_{j}, f_{i j}\right)\right) \quad\left(\text { resp. }{\underset{\longrightarrow}{\lim }}_{j} X_{j}=F\left(\left(X_{j}, f_{i j}\right)\right)\right.
$$

for any such system ( $X_{i}, f_{i j}$ ).
Proof. This follows by applying Proposition 2.14.
Remark 2.23. In fact, Proposition 2.20 shows how to define $\npreceq F$ for any contravariant functor, and $\underline{l i m} F$ for any functor using suitable adjoints (if they exist). Since any commutative diagram can be viewed as a functor, this will allow us to consider limits of diagrams. We will not do this very often, but occasionally, this point of view will be useful.

## Example 2.24.

(1). If $J$ is any set and $\leqslant$ is equality (i.e. $x \leqslant y$ if and only if $x=y$ ), then the projective limit of $X_{j}$ (if it exists) is called the product and denoted by $\prod_{j \in J} X_{j}$. It comes with morphisms $p_{j}: \prod_{j \in J} X_{j} \rightarrow X_{j}$ for all $j \in J$. It is characterised by the universal property that given morphisms $g_{j}: Y \rightarrow X_{j}$ for some object $Y$, there is a unique morphism $g: Y \rightarrow \prod_{j \in J} X_{j}$ such that $p_{j} \circ g=g_{j}$. The morphism $g$ is often denoted as a tuple $\left(g_{j}\right)_{j \in J}$.

In particular, whenever one is given morphisms $f_{j}: X_{j} \rightarrow Y_{j}$, one may define $g_{j}: \prod_{j \in J} X_{j} \rightarrow Y_{j}$ by $g_{j}=f_{j} \circ p_{j}$. The morphism $\prod_{j \in J} X_{j} \rightarrow \prod_{j \in J} Y_{j}$ obtained by the universal property is often denoted by $\prod_{j \in J} f_{j}$. If $X_{j}=Y$ for all $j \in J$, one may also construct $\delta_{Y}: Y \rightarrow \prod_{j \in J} Y$ by requiring $p_{j} \circ \delta_{Y}=\operatorname{id}_{Y}$ for all $j \in J$. This is the so-called diagonal morphism.
(2). A special case of product is in given by the case of $J=\varnothing$. The empty product (if it exists) is usually denoted by $*$. It is characterised by the property that for any object $Y$, there is a unique morphism $Y \rightarrow *$. For this reason, the empty product is also called the terminal or final object of $\mathcal{C}$.

The category Sets has a terminal object, given by a singleton set. The category of modules over a ring also has a terminal object, the trivial module.
(3). If $J$ is any set and $\leqslant$ is equality, then the inductive limit of $X_{j}$ (if it exists) is called the coproduct and denoted by $\coprod_{j \in J} X_{j}$. It comes with morphisms $i_{j}: X_{j} \rightarrow X$ and is characterised by the universal property that given morphisms $g_{j}: X_{j} \rightarrow Y$ for some object $Y$, there is a unique morphism $g: \coprod_{j \in J} X_{j} \rightarrow Y$ such that $g \circ i_{j}=g_{j}$.

In the category Sets, coproducts exist, and are given by disjoint unions. In the category of modules over a ring, coproduct also exist, and are given by direct sums. In this category, finite products and coproducts of the same family of objects are isomorphic.

Corollary 2.25. The Yoneda Embedding h commutes with projective limits. That is, if $X=\lim _{j} X_{j}$ in $\mathcal{C}$, then $X(-)=\lim _{j} X_{j}(-)$ in $\mathcal{C}^{\vee}$.
Proof. Let $F: J^{o p} \rightarrow \mathcal{C}$ be the functor corresponding to the projective system, and $\Delta_{\mathcal{C}}: \mathcal{C} \rightarrow \mathcal{C}^{J^{\text {op }}}$ the diagonal functor for $\mathcal{C}$. If $X=\varliminf_{\varliminf_{j}} X_{j}$, then

$$
\operatorname{Hom}_{\mathcal{C}^{J o p}}\left(\Delta_{\mathcal{C}}(Y), F\right)=\operatorname{Hom}_{\mathcal{C}}(Y, X)=X(Y)=\operatorname{Hom}_{\mathcal{C}} \vee(Y(-), X(-))
$$

by Proposition 2.20 and the Yoneda Lemma.
On the other hand, let $\Delta_{\mathcal{C}} \vee \mathcal{C}^{\vee} \rightarrow\left(\mathcal{C}^{\vee}\right)^{J o p}$ be the diagonal functor for $\mathcal{C}^{\vee}$. Then $\Delta_{\mathcal{C}}{ }^{\vee}(Y(-))_{j}=Y(-)=\left[\Delta_{\mathcal{C}}(Y)_{j}\right](-)$, and similarly for morphisms. That is, the Yoneda Embedding intertwines diagonal functors. Thus, the Yoneda Lemma shows for any $Y \in \mathrm{Ob} \mathcal{C}$,

$$
\operatorname{Hom}_{\left(\mathcal{C}^{\vee}\right)^{\text {Jop }}}\left(\Delta_{\mathcal{C}^{\vee}}(Y(-)), X(-)\right)=\operatorname{Hom}_{\mathcal{C}}\left(\Delta_{\mathcal{C}}(Y), X\right)
$$

Thus, $X(-)$ represents the functor $\operatorname{Hom}_{\left(\mathcal{C}^{\vee}\right)^{\text {Jop }}}\left(\Delta_{\mathcal{C}^{\vee}-}, X(-)\right)$. In view of Proposition 2.20, this proves our claim.
Remark 2.26.
(1). The Yoneda embedding does not commute with inductive limits in general. Indeed, if $\mathcal{C}$ is the category of modules over a ring $R$, then coproducts exist and are given by direct sums of modules. However, coproducts in $\mathcal{C}^{\vee}$ are given by disjoint unions.
(2). Corollary 2.25 provides a general recipe for computing projective limits, and indeed of proving their existence: If $\lim _{j} X_{j}(-)$ exists in $\mathcal{C}^{\vee}$ and the underlying object (i.e. set-valued contavariant functor) is represented by $X$, then $X=\lim _{j} X_{j}$.

Indeed, it is also clear that for any $Y \in \mathrm{Ob} \mathrm{\mathcal{C}}$, the value of $\lim _{\leftrightarrows_{j}} X_{j}(-)$ (if it exists) has to be given by $\lim _{\leftrightarrows} X_{j}(Y)$, where the limit is computed in Sets. Thus, the construction of projective limits can be reduced their construction in Sets, together with a question of representability.
Proposition 2.27. Let $(J, \leqslant)$ be a preordered set.
(1). Let $\left(X_{i}, f_{i j}\right)$ be a projective system in Sets, indexed by J. Define $X \subset \prod_{j \in J} X_{j}$ by

$$
x \in X \quad: \Leftrightarrow \quad \forall i \leqslant j: x_{i}=f_{i j}\left(x_{j}\right)
$$

and $f_{i}: X \rightarrow X_{i}$ by $f_{i}(x)=x_{i}$. Then $X=\lim _{j} X_{j}$.
(2). Let $\left(X_{i}, f_{i j}\right)$ be an inductive system in Sets, indexed by J. Define $X=\coprod_{j \in J} X_{j} / \sim$ where $\sim$ is the transitive hull of the relation $\approx$ given for $x \in X_{i}, y \in X_{j}$ by

$$
x \approx y \quad: \Leftrightarrow \quad \exists k: i \leqslant k, j \leqslant k, f_{i k}(x)=f_{j k}(y)
$$

Let $f_{i}: X_{i} \rightarrow X$ map $x \in X_{i}$ to its equivalence class. Then $X=\underline{\lim }_{j} X_{j}$.

Proof. This is left as an exercise. Observe that if $\leqslant$ is an order in (2), then it is not necessary to take transitive hulls.

With a view towards applications to supermanifolds, we introduce the following concept.
Definition 2.28. Let $\mathcal{C}$ be a category with finite products (i.e. the product of any finite set of objects exists). A group object in $\mathcal{C}$ is a quadruple ( $G, m, i, e$ ) where $G \in \mathrm{Ob} \mathcal{C}$, and $m: G \times G \rightarrow G, i: G \rightarrow G$ and $e: * \rightarrow G$ are morphisms, such that the following diagrams commute:


Here, $G \rightarrow *$ is the unique morphism to the terminal object, $G \rightarrow G \times *$ is the product of id with this morphism, and similarly for $G \rightarrow * \times G$.

A morphism of group objects $f:(G, m, i, e) \rightarrow\left(G^{\prime}, m^{\prime}, i^{\prime}, e^{\prime}\right)$ is a morphism $f: G \rightarrow G^{\prime}$ such that

$$
f \circ m=m^{\prime} \circ(f \times f), f \circ i=i^{\prime} \circ f \text { and } f \circ e=e^{\prime} .
$$

One thus obtains a category of groups objects in $\mathcal{C}$.
By abuse of notation, one usually denotes by $G$ a group object in $\mathcal{C}$, the morphisms $m, i$ and $e$ being understood.
Corollary 2.29. Let $\mathcal{C}$ be a category with finite products. The Yoneda Embedding induces a fully faithful functor from the category of group objects in $\mathcal{C}$ to the category of contravariant functors on $\mathcal{C}$ with values in groups.
Proof. By Corollary 2.25, the Yoneda Embedding induce a fully faithful functor between the category of group objects in $\mathcal{C}$ and $\mathcal{C}^{\vee}$. Let $F \in \mathcal{C}^{\vee}$. Then by what we have remarked about projective limits in $\mathcal{C}^{\vee}, F$ defines a group object in $\mathcal{C}^{\vee}$ if and only if it defines a functor with values in groups.

Remark 2.30. For a group object $G$, the group structure on $G(S)$ is given by setting, for $s, t \in_{S} G$, st $=m(s, t), s^{-1}=i(s)$, and $1_{G(S)}=e\left(*_{S}\right)$ where $*_{S}$ is the unique morphism $S \rightarrow$.

By Corollary 2.29, one shows, using standard facts on groups, that $i$ and $e$ are uniquely determined by $m$. Thus, a morphism $f: G \rightarrow G^{\prime}$ between the underlying objects of group objects in $\mathcal{C}$ is a morphism of group objects if and only if $f \circ m=m^{\prime} \circ(f \times f)$, if and only if $f(s t)=f(s) f(t)$ for all $S \in \mathrm{ObC}$ and all $s, t \in_{S} G$. The equations $f\left(s^{-1}\right)=f(s)^{-1}$ and $f(1)=1$ are automatically verified.

We end the section by a discussion of iterated limits.
Proposition 2.31. Let $I$ and $J$ be preordered sets. Let $X_{i j} \in \mathrm{ObC}$ and $f_{i i^{\prime}}^{j}: X_{i^{\prime} j} \rightarrow X_{i j}$ resp. $g_{j j^{\prime}}^{i}: X_{i j^{\prime}} \rightarrow X_{i j}$ for $i \leqslant i^{\prime}$ resp. $j \leqslant j^{\prime}$ satisfy:
(1). For every $j \in J,\left(X_{i j}, f_{i i^{\prime}}^{j}\right)$ is a projective system indexed by $I$;
(2). for every $i \in I,\left(X_{i j}, g_{j j^{\prime}}^{i}\right)$ is a projective system indexed by $J$;
(3). for all $i \leqslant i^{\prime}, j \leqslant j^{\prime}$, we have

$$
f_{i i^{\prime}}^{j} \circ g_{j j^{\prime}}^{i^{\prime}}=g_{j j^{\prime}}^{i} \circ f_{i i^{\prime}}^{j^{\prime}}
$$

Whenever $\lim _{\leftarrow} \lim _{\hookleftarrow} X_{i j}$ and $\lim _{\varlimsup_{j}} \lim _{i} X_{i j}$ exist, they are canonically isomorphic. The analoguous statement holds for inductive limits.

Proof. Let $K=I \times J$, preordered by the relation

$$
(i, j) \leqslant\left(i^{\prime}, j^{\prime}\right) \quad: \Longleftrightarrow \quad i \leqslant i^{\prime} \text { and } j \leqslant j^{\prime}
$$

For $k=(i, j) \in K$, let let $X_{k}=X_{i j}$. For $k=(i, j) \leqslant k^{\prime}=\left(i^{\prime}, j^{\prime}\right)$,

$$
h_{k k^{\prime}}=f_{i i^{\prime}}^{j} \circ g_{j j^{\prime}}^{i^{\prime}}=g_{j j^{\prime}}^{i} \circ f_{i i^{\prime}}^{j^{\prime}}: X_{k^{\prime}}=X_{i^{\prime} j^{\prime}} \rightarrow X_{i j}=X_{k}
$$

If $k=(i, j) \leqslant k^{\prime}=\left(i^{\prime}, j^{\prime}\right) \leqslant k^{\prime \prime}=\left(i^{\prime \prime}, j^{\prime \prime}\right)$, then

$$
\begin{aligned}
h_{k k^{\prime}} \circ h_{k^{\prime} k^{\prime \prime}} & =f_{i i^{\prime}}^{j} \circ g_{j j^{\prime}}^{i^{\prime}} \circ g_{j^{\prime} j^{\prime \prime}}^{i^{\prime}} \circ f_{i^{\prime} i^{\prime \prime}}^{j^{\prime \prime}}=f_{i i^{\prime}}^{j} \circ g_{j j^{\prime \prime}}^{i^{\prime}} \circ f_{i^{\prime} i^{\prime \prime}}^{j^{\prime \prime}} \\
& =f_{i i^{\prime}}^{j} \circ f_{i^{\prime} i^{\prime \prime}}^{j} \circ g_{j j^{\prime \prime}}^{i^{\prime \prime}}=f_{i i^{\prime \prime}}^{j} \circ g_{j j^{\prime \prime}}^{i^{\prime \prime}}=h_{k k^{\prime \prime}},
\end{aligned}
$$

so $\left(X_{k}, h_{k k^{\prime}}\right)$ is a projective system indexed over $K$.
It will be sufficient to show that if ${\underset{\zeta i m}{i}}^{\lim _{j}} X_{i j}$ exists, then it is a projective limit of $\left(X_{k}, h_{k k^{\prime}}\right)$. Indeed, exchanging the role of $I$ and $J$, the assertion will follow.

Assume that $X=\lim _{\leftarrow} \lim _{\leftarrow} X_{i j}$ exists. For $k=(i, j)$, we define the morphism $h_{k}: X \rightarrow X_{k}=X_{i j}$ as the composition of

$$
f_{i}: X \rightarrow X_{i}=\lim _{\gtrless_{j}} X_{i j} \quad \text { and } \quad g_{j}^{i}: X_{i} \rightarrow X_{i j}
$$

Consider also the morphisms $f_{i i^{\prime}}: X_{i^{\prime}} \rightarrow X_{i}$ uniquely determined by

$$
g_{j}^{i} \circ f_{i i^{\prime}}=f_{i i^{\prime}}^{j} \circ g_{j}^{i^{\prime}}
$$

By definition, $g_{j j^{\prime}}^{i} \circ g_{j^{\prime}}^{i}=g_{j}^{i}$ and $f_{i i^{\prime}} \circ f_{i^{\prime}}=f_{i}$.
For $k=(i, j) \leqslant k^{\prime}=\left(i^{\prime}, j^{\prime}\right)$,

$$
\begin{aligned}
h_{k k^{\prime}} \circ h_{k^{\prime}} & =f_{i i^{\prime}}^{j} \circ g_{j j^{\prime}}^{i^{\prime}} \circ g_{j^{\prime}}^{i^{\prime}} \circ f_{i^{\prime}}=f_{i i^{\prime}}^{j} \circ g_{j}^{i^{\prime}} \circ f_{i^{\prime}} \\
& =g_{j}^{i} \circ f_{i i^{\prime}} \circ f_{i^{\prime}}=g_{j}^{i} \circ f_{i}=h_{k}
\end{aligned}
$$

Now, assume that we are given morphisms $\varphi_{k}: Y \rightarrow X_{k}$ such that $h_{k k^{\prime}} \circ \varphi_{k^{\prime}}=\varphi_{k}$ for $k \leqslant k^{\prime}$. For $i \in I$, define $\varphi_{i}: Y \rightarrow X_{i}$ to be the unique morphism such that for all $j \in J, g_{j}^{i} \circ \varphi_{i}=\varphi_{k}$ where $k=(i, j)$. Define $\varphi_{i}: Y \rightarrow X$ to be the unique morphism such that $f_{i} \circ \varphi=\varphi_{i}$ for all $i \in I$. Then for $k=(i, j)$,

$$
h_{k} \circ \varphi=g_{j}^{i} \circ f_{i} \circ \varphi=g_{j}^{i} \circ \varphi_{i}=\varphi_{k}
$$

Let $\psi: Y \rightarrow X$ be any morphism such that $h_{k} \circ \psi=\varphi_{k}$. Then for all $i \in I$ and $j \in J$,

$$
g_{j}^{i} \circ\left(f_{i} \circ \psi\right)=h_{k} \circ \psi=\varphi_{k}
$$

so $\varphi_{i}=f_{i} \circ \psi$ for all $i \in I$ and it follows that $\varphi=\psi$. Thus, $X$ is indeed a projective limit of $\left(X_{k}, h_{k k^{\prime}}\right)$, and this proves the claim.

## 3. Sheaves

### 3.1. Presheaves and sheaves.

Definition 3.1. Let $X$ be a topological space and $\mathcal{C}$ a category. A presheaf on $X$ with values in $\mathcal{C}$ is a map $U \mapsto \mathcal{F}(U)$ from open subsets of $X$ to objects of $\mathcal{C}$, together with the data of morphisms $\varrho_{V U}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for any open sets $V \subset U$. It is customary and suggestive to call $\varrho_{V U}$ the restriction from $U$ to $V$. If $\mathcal{C}$ is a subcategory of Sets, then one writes $\left.f\right|_{V}$ for $\varrho_{V U}(f)$. Usually, the restriction morphisms of a presheaves are understood, although they are strictly speaking part of the data.

A morphism of presheaves on $X, \varphi: \mathcal{F} \rightarrow \mathcal{F}^{\prime}$ (say), is a collection of morphisms $\varphi_{U}: \mathcal{F}(U) \rightarrow \mathcal{F}^{\prime}(U)$, for any open $U \subset X$, such that all of the following diagrams commute:


It is clear how to compose morphisms of presheaves. With this composition and the obvious identity morphisms, we obtain the category $\operatorname{Presh}_{\mathcal{C}}(X)$ of all presheaves on $X$ with values in $\mathcal{C}$.

In what follows, we will usually consider presheaves with values in ${ }_{R} \mathrm{Mod}$, for some fixed ring $R$; for instance, if $R=\mathbb{Z}$, then we will be dealing with sheaves of Abelian groups, and if $R=k$ is a field, we will be dealing with sheaves of vector spaces. The category of presheaves in ${ }_{R} \operatorname{Mod}$ on $X$ will be denoted by Presh $(X)$. More generally, the constructions we will perform work with only notational changes for presheaves with values in any Abelian, and even, with some restrictions, in any additive category.

Example 3.2. If $U \subset X$ is open and $\mathcal{F}$ is a presheaf on $X$, then there is a presheaf $\left.\mathcal{F}\right|_{U}$ on $U$ defined by $\left.\mathcal{F}\right|_{U}(V)=\mathcal{F}(V)$ for all open $V \subset U$. It is called the restriction of $\mathcal{F}$ to $U$.

For any open $U \subset X$, we let $\Gamma(U, \mathcal{F})=\mathcal{F}(U)$, the set of sections over $U$. By the definition of morphisms, $\Gamma(U, \cdot)$, which maps $\mathcal{F}$ to $\Gamma(U, \mathcal{F})=\mathcal{F}(U)$ and $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ to $\Gamma(U, \varphi)=\varphi_{U}$, is a functor.

Definition 3.3. Let $X$ be topological space and $\left(U_{i}\right)_{i \in I}$ an open cover of $X$. One denotes $U_{i j}=U_{i} \cap U_{j}$ and $U_{i j k}=U_{i} \cap U_{j} \cap U_{k}$ for $i, j, k \in I$.

Let $\mathcal{F} \in \operatorname{Presh}(X)$. Then $\mathcal{F}$ is called a sheaf if the following two axioms are fulfilled for any open subset $U \subset X$ and any open cover $\left(U_{i}\right)$ of $U$ :
(i). If $f, g \in \mathcal{F}(U)$ are such that $\left.f\right|_{U_{i}}=\left.g\right|_{U_{i}}$ for all $i \in I$, then $f=g$.
(ii). Given $f_{i} \in \mathcal{F}\left(U_{i}\right), i \in I$, such that $\left.f_{i}\right|_{U_{i j}}=\left.f_{j}\right|_{U_{i j}}$ for all $i, j \in I$, there is $f \in \mathcal{F}(U)$ such that $\left.f\right|_{U_{i}}=f_{i}$ for all $i \in I$.
The full subcategory of $\operatorname{Presh}(X)$ formed by the sheaves on $X$ is denoted by $\operatorname{Sh}(X)$.

Remark 3.4. (1). Sheaf axiom (i) implies that $\mathcal{F}(\varnothing)=0$. Indeed, $\varnothing$ is covered by the empty family $\left(U_{i}\right)_{i \in I}$ where $I=\varnothing$. Hence, if $f \in \mathcal{F}(\varnothing)$, then $\left.f\right|_{U_{i}}=0$ for all $i \in I$, simply because $I$ is empty. Then (i) implies that
$f=0$. More generally, if one takes presheaves with values in an arbitrary subcategory of sets, then sheaf axioms (i) and (ii) imply that $\mathcal{F}(\varnothing)$ is a singleton set. (Thus for some such categories, there are no sheaves.)
(2). The sheaf axioms are equivalent to the statement that for any open $U \subset X$ and any open cover $\mathcal{U}$ of $U$, stable by finite intersections, the morphism $\mathcal{F}(U) \rightarrow \lim _{V \in \mathcal{U}} \mathcal{F}(V)$ given by restriction is an isomorphism. This may be taken as the definition of a sheaf with values in a category which has projective limits. In this formulation, it follows that $\mathcal{F}(\varnothing)$ is a terminal object (as the projective limit over an empty set).

Many constructions one wishes to apply to sheaves initially only produce presheaves. Thus, one needs a construction of sheaves from presheaves. We shall now examine such a construction.
3.5. Let $\mathcal{F}$ be a presheaf on $X$ and $x \in X$. We define

$$
\begin{equation*}
\mathcal{F}_{x}=\lim _{\longrightarrow} \mathcal{F}(U) \tag{3.1}
\end{equation*}
$$

where $U$ runs over all open neighbourhoods of $x$. The Abelian group $\mathcal{F}_{x}$ is called the stalk of $\mathcal{G}_{x}$, and the image of $f$ in $\mathcal{F}_{x}$ is denoted $f_{x}$ and called the germ of $f$ at $x$.

This construction is functorial. In particular, for any morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ of presheaves, we obtain morphisms of the stalks $\varphi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$.
3.6. Let $\mathcal{F}$ be a presheaf on $X$. For $U \subset X$ open, let $\mathcal{F}^{+}(U)$ be the subset of $\prod_{x \in X} \mathcal{F}_{x}$ consisting of all $f$ such that for all $x \in U$, there exists an open open neighbourhood $V \subset U$ and $\tilde{f} \in \mathcal{F}(V)$ such that $f(y)=\tilde{f}_{y}$ for all $y \in V$.

Then it is clear that $\mathcal{F}^{+}$defines in a natural way a presheaf. Moreover, there is a canonical morphism $\mathcal{F} \rightarrow \mathcal{F}^{+}$. Clearly, $\mathcal{F}_{x}=\mathcal{F}_{x}^{+}$for all $x \in X$.
Proposition 3.7. Let $\mathcal{F}$ be a presheaf on $X$.
(i). $\mathcal{F}^{+}$is a sheaf, called the sheafification of $\mathcal{F}$.
(ii). $(-)^{+}$defines a functor $\operatorname{Presh}(X) \rightarrow \operatorname{Sh}(X)$, and the canonical morphism gives a natural transformation $\theta$ : id $\rightarrow(-)^{+}$.
(iii). $\mathcal{F} \rightarrow \mathcal{F}^{+}$is an isomorphism if and only if $\mathcal{F}$ is a sheaf.
(iv). One has a natural isomorphism $\operatorname{Hom}(\mathcal{F}, \mathcal{G})=\operatorname{Hom}\left(\mathcal{F}^{+}, \mathcal{G}\right)$ as $\mathcal{G}$ varies over $\operatorname{Sh}(X)$, and $\mathcal{F}$ varies over $\operatorname{Presh}(X)$. In other words, sheafification is left adjoint to the inclusion $\operatorname{Sh}(X) \rightarrow \operatorname{Presh}(X)$.
Proof. Statements (i) and (ii) are obvious.
Proof of (iii). The map $\mathcal{F}(U) \rightarrow \prod_{x \in U} \mathcal{F}_{x}: f \mapsto\left(f_{x}\right)$ is clearly injective if and only sheaf axiom (i) holds on $U$. Let $\mathcal{F}$ be a sheaf and $f \in \mathcal{F}^{+}(U)$. For $x \in U$, there is an open neighbourhood $U_{x} \subset U$ of $x$ and $\tilde{f}^{x} \in \mathcal{F}\left(U_{x}\right)$ such that $\left(\tilde{f}^{x}\right)_{y}=f(\underset{\sim}{y})$ for all $\underset{\sim}{y} \in U_{x}$. Then $\left(\tilde{f}^{x}\right)_{z}=\left(\tilde{f}^{y}\right)_{z}$ for all $z \in U_{x} \cap U_{y}$, and it follows that $\left.\tilde{f}^{x}\right|_{U_{x y}}=\left.\tilde{f}^{y}\right|_{U_{x y}}$. By sheaf axiom (ii), there is $\tilde{f} \in \mathcal{F}(U)$ such that $\left.\tilde{f}\right|_{U^{x}}=f^{x}$. In particular, $\tilde{f}_{x}=f(x)$. The converse follows similarly.

Proof of (iv). We have that $\theta_{\mathcal{G}}: \mathcal{G} \rightarrow \mathcal{G}^{+}$is an isomorphism. Consider

$$
\operatorname{Hom}(\mathcal{F}, \mathcal{G}) \underset{\Psi}{\stackrel{\Phi}{\rightleftarrows}} \operatorname{Hom}\left(\mathcal{F}^{+}, \mathcal{G}\right)
$$

given by $\Phi(\varphi)=\theta_{\mathcal{G}}^{-1} \circ \varphi^{+}$and $\Psi(\psi)=\psi \circ \theta_{\mathcal{F}}$. By naturality of $\theta$ (part (ii)), we have $\theta_{\mathcal{G}} \circ \varphi=\varphi^{+} \circ \theta_{\mathcal{F}}$, and this proves that $\Psi \circ \Phi=\mathrm{id}$. Moreover,
$\Phi(\Psi(\psi)) \circ \theta_{\mathcal{F}}=\psi \circ \theta_{\mathcal{F}}$ for any $\psi \in \operatorname{Hom}\left(\mathcal{F}^{+}, \mathcal{G}\right)$. The latter statement implies that $\Psi \circ \Phi=$ id.

Indeed, let $\psi \circ \theta_{\mathcal{F}}=\psi^{\prime} \circ \theta_{\mathcal{F}}$ where $\psi, \psi^{\prime}: \mathcal{F}^{+} \rightarrow \mathcal{G}$. For $f \in \mathcal{F}^{+}(U)$ and $x \in U$, choose open neighbourhoods $U_{x} \subset U$ and $\tilde{f}^{x} \in \mathcal{F}\left(U_{x}\right)$ such that $\left(\tilde{f}^{x}\right)_{y}=f(y)$ for all $y \in U_{x}$. Then $\left.f\right|_{U_{x}}=\theta_{\mathcal{F}, U_{x}}\left(\tilde{f}^{x}\right)$, so

$$
\left.\psi_{U}(f)\right|_{U_{x}}=\psi_{U_{x}}\left(\left.f\right|_{U_{x}}\right)=\left(\psi \circ \theta_{\mathcal{F}}\right)_{U_{x}}\left(\tilde{f}^{x}\right)=\left(\psi^{\prime} \circ \theta_{\mathcal{F}}\right)_{U_{x}}\left(\tilde{f}^{x}\right)=\left.\psi_{U}^{\prime}(f)\right|_{U_{x}} .
$$

By the first sheaf axiom, $\psi_{U}(f)=\psi_{U}^{\prime}(f)$.
Example 3.8. Let $A$ be an $R$-module and $A_{X}$ be the sheafification of $U \mapsto A$. Then $A_{X}$ is called the constant sheaf with stalk $A$. For $x \in X, A_{X, x} \cong A$.

Corollary 3.9. Let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on $X$. Then $\varphi$ is an isomorphism if and only if for each $x \in X, \varphi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ is.

Proof. Necessity is obvious by the functoriality of the stalk construction.
Conversely, assume that $\varphi_{x}: \mathcal{F}_{x} \rightarrow \mathcal{G}_{x}$ is an isomorphism for any $x \in X$.
We obtain an isomorphism

$$
\tilde{\varphi}=\left(\varphi_{x}\right): \prod_{x \in X} \mathcal{F}_{x} \rightarrow \prod_{x \in X} \mathcal{G}_{x} .
$$

Consider the inverse $\tilde{\psi}=\tilde{\varphi}^{-1}$. Let $U \subset X$ be open and $g \in \mathcal{G}(U)$. For $x \in X$, consider $f(x)=\tilde{\psi}\left(g_{x}\right) \in \mathcal{F}_{x}$. Let $U_{x} \subset U$ be an open neighbourhood of $x$ and $f^{x} \in \mathcal{F}\left(U_{x}\right)$ such that $\left(f^{x}\right)_{x}=f(x)$. Then $\varphi_{U_{x}}\left(f^{x}\right)_{x}=\varphi_{x}(f(x))=g_{x}$, so there exists an open neighbourhood $V_{x} \subset U_{x}$ of $x$ such that $\varphi_{V_{x}}\left(\left.f^{x}\right|_{V_{x}}\right)=$ $\left.\varphi_{U_{x}}\left(f^{x}\right)\right|_{V_{x}}=\left.g\right|_{V_{x}}$. In particular, $\varphi_{y}\left(f^{x}\right)_{y}=g_{y}$ for all $y \in V_{x}$, and this implies $\left(f^{x}\right)_{y}=f(y)$ for all $y \in V_{x}$, by construction. Thus, $f=(f(x))_{x \in U}$ is contained in $\mathcal{F}^{+}(U)$. Defining $\psi_{U}(g)=f$, we get a morphism $\mathcal{G}=\mathcal{G}^{+} \rightarrow \mathcal{F}^{+}$ which is inverse to $\varphi^{+}$. By Proposition 3.7 (iv), this proves the claim.
Proposition 3.10. Let $\left(\mathcal{F}_{j}, \varphi_{i j}\right)$ be a projective system of sheaves on $X$, indexed by an preordered set $J$. Then

$$
\mathcal{F}(U)=\lim _{\leftrightarrows} \mathcal{F}_{j}(U)
$$

defines a sheaf $\mathcal{F}$, and it is the projective limit of the $\left(\mathcal{F}_{j}, \varphi_{i j}\right)$ in $\operatorname{Sh}(X)$. Hence, we denote $\mathcal{F}=\lim _{j} \mathcal{F}_{j}$ and call this the projective limit sheaf.
Proof. Recall that $\mathcal{F}$ is a sheaf if and only if $\mathcal{F}(U)=\lim _{V \in \mathcal{U}} \mathcal{F}(V)$ for any open $U \subset X$ and any open cover $\mathcal{U}$ which is stable under finite intersections. Thus, the assertion follows from Proposition 2.31.
Remark 3.11. Let $\left(\mathcal{F}_{j}, \varphi_{i j}\right)$ be a projective system of sheaves. If $x \in X$, then there is a canonical morphism $\left(\lim _{j} \mathcal{F}_{j}\right)_{x} \rightarrow \varliminf_{j} \mathcal{F}_{j, x}$. In general, it is not an isomorphism.
3.2. Direct and inverse image sheaves. We will now introduce a number of operations on sheaves, beginning with the direct and inverse image.

Definition 3.12. Let $f: X \rightarrow Y$ be a continuous map, $\mathcal{F} \in \operatorname{Sh}(X)$, and $\mathcal{G} \in \operatorname{Sh}(Y)$. The direct image $f_{*} \mathcal{F}$ of $\mathcal{F}$ under $f$ is the sheaf on $Y$ defined by the equation $f_{*} \mathcal{F}(V)=\mathcal{F}\left(f^{-1}(V)\right)$, for any open $V \subset Y$. Morphisms $\varphi: \mathcal{F} \rightarrow \mathcal{F}^{\prime}$ are mapped to morphisms $f_{*}(\varphi): f_{*} \mathcal{F} \rightarrow f_{*} \mathcal{F}^{\prime}$, given by $\left(f_{*}(\varphi)\right)_{V}=\varphi_{f^{-1}(V)}$.

The inverse image $f^{-1} \mathcal{F}$ of $\mathcal{G}$ under $g$ is the sheaf on $X$ defined as the sheafification of the presheaf $U \mapsto{\underset{\longrightarrow}{\lim }}_{V f(U)} \mathcal{G}(V)$, where $V$ runs over the open neighbourhoods of $f(U)$ in $Y$.

As a matter of terminology, it is convenient to refer to the image of $g \in \mathcal{G}(V)$ in the injective limit as the germ of $g$ on $f(V)$.

Morphisms $\psi: \mathcal{G} \rightarrow \mathcal{G}^{\prime}$ are mapped to morphisms $f^{-1}(\psi): f^{-1} \mathcal{G} \rightarrow f^{-1} \mathcal{G}^{\prime}$, given by $\left(f^{-1}(\psi)\right)_{U}={\underset{\longrightarrow}{l}}_{V \supset f(U)} \psi_{V}$.

We obtain functors

$$
f_{*}: \operatorname{Sh}(X) \rightarrow \operatorname{Sh}(Y) \text { and } f^{-1}: \operatorname{Sh}(Y) \rightarrow \operatorname{Sh}(X)
$$

such that $(g \circ f)_{*}=g_{*} \circ f_{*}$ and $(g \circ f)^{-1}=f^{-1} \circ g^{-1}$ whenever $g: Y \rightarrow Z$ is continuous. (The second equality is not entirely obvious, but will follow from the first in view of Proposition 3.15 below.)

Example 3.13. Let $j: U \rightarrow X$ be the inclusion of the open subset $U \subset X$. Then $j^{-1} \mathcal{F}=\left.\mathcal{F}\right|_{U}$ for any sheaf $\mathcal{F}$ on $X$. In particular, by their definition, $j^{-1}$ commutes with projective limits.

Example 3.14. Let $f: X \rightarrow Y$ be continuous, $A$ an Abelian group. Then $f^{-1} A_{Y}=A_{X}$.

Proposition 3.15. Let $f: X \rightarrow Y$ be continuous and $\mathcal{R}$ be a sheaf of rings on $Y$. There is a natural isomorphism

$$
\operatorname{Hom}\left(\mathcal{F}, f_{*} \mathcal{G}\right) \cong \operatorname{Hom}\left(f^{-1} \mathcal{F}, \mathcal{G}\right)
$$

as $\mathcal{F}$ runs over $\operatorname{Sh}(Y)$ and $\mathcal{G}$ runs over $\operatorname{Sh}(X)$.
3.16. Before we delve into the proof, let us introduce two important natural transformations which will make the statement of the proposition meaningful.

Let $\tilde{\mathcal{F}}(U)={\underset{\longrightarrow}{\lim }}_{V \supset f(U)} \mathcal{F}(V)$, so that $f^{-1} \mathcal{F}=\tilde{\mathcal{F}}^{+}$. We exhibit a morphism $\alpha_{\mathcal{G}} \in \operatorname{Hom}\left(f^{-1} f_{*} \mathcal{G}, \mathcal{G}\right)=\operatorname{Hom}\left(\widetilde{f_{*} \mathcal{G}}, \mathcal{G}\right)$ by defining

$$
\alpha_{\mathcal{G}, U}: \lim _{V \overrightarrow{\supset f(U)}} f_{*} \mathcal{G}(V)=\lim _{V \overrightarrow{\supset f(U)}} \mathcal{G}\left(f^{-1}(V)\right) \rightarrow \mathcal{G}(U)
$$

as the limit of all restriction maps. Since morphisms commute with restriction, this defines a natural transformation $\alpha: f^{-1} f_{*} \rightarrow$ id of functors in $\operatorname{Sh}(X)$.

Next, we construct $\beta_{\mathcal{F}} \in \operatorname{Hom}\left(\mathcal{F}, f_{*} f^{-1} \mathcal{F}\right)$. For any open $V \subset Y$, we have $V \supset f\left(f^{-1}(V)\right)$, so there is a canonical morphism

$$
\mathcal{F}(V) \rightarrow \lim _{W \supset f\left(f^{-1}(V)\right)} \mathcal{F}(W)=\tilde{\mathcal{F}}\left(f^{-1}(V)\right) .
$$

Let $\beta_{\mathcal{F}, V}$ be the composition of this map with $\theta_{\tilde{\mathcal{F}}, f^{-1}(V)}$. Manifestly, this defines a natural transformation $\beta:$ id $\rightarrow f_{*} f^{-1}$ in $\operatorname{Sh}(Y)$.

Proof of Proposition 3.15. According to Exercise 2.12, we need to prove that the following composites are the identity:

$$
f^{-1} \mathcal{F} \xrightarrow{f^{-1}\left(\beta_{\mathcal{F}}\right)} f^{-1} f_{*} f^{-1} \mathcal{F} \xrightarrow{\alpha_{f-1} \mathcal{F}} f^{-1} \mathcal{F}
$$

and

$$
f_{*} \mathcal{G} \xrightarrow{\beta_{f_{*} \mathcal{G}}} f_{*} f^{-1} f_{*} \mathcal{G} \xrightarrow{f_{*}\left(\alpha_{\mathcal{G}}\right)} f_{*} \mathcal{G}
$$

Let us consider the first composite. By Proposition 3.7, it is sufficient to compute on the level of the presheaf $\tilde{\mathcal{F}}$, the sheafification of which is $f^{-1} \mathcal{F}$. Let $U \subset X$ be open, and $h \in \tilde{\mathcal{F}}(U)={\underset{\longrightarrow}{\lim }}_{V \supset f(U)} \mathcal{F}(V)$. There exist an open $V \subset Y$ such that $V \supset f(U)$ and $h_{V} \in \mathcal{F}(V)$ such that $h$ is the germ of $g_{V}$ on $f(U)$.

Then $f^{-1}\left(\beta_{\mathcal{F}}\right)_{U}(g)=\beta_{\mathcal{F}, V}\left(h_{V}\right)$ is $\theta_{\tilde{\mathcal{F}}, f^{-1}(V)}$ applied to the germ of $h_{V}$ on the subset $f\left(f^{-1}(V)\right) \subset Y$. If we apply $\alpha_{f^{-1} \mathcal{F}, U}$ to this, we obtain the germ of $h_{V}$ on the set $f\left(f^{-1}(f(U))\right)=f(U)$, namely, $h$. Thus, the first composite is indeed the identity. A similar computation also establishes this fact in the second case.

Example 3.17. Let $j: U \rightarrow X$ be the inclusion of an open subset. Then the natural transformation id $\rightarrow j_{*} j^{-1}$ is an equivalence.

This simple observation can be used to glue sheaves by considering projective limits.

Proposition 3.18. Let $\mathcal{U}=\left(U_{i}\right)$ be an open cover of $X, \mathcal{F}_{i} \in \operatorname{Sh}\left(U_{i}\right), i \in I$, and $\varphi_{i j}:\left.\left.\mathcal{F}_{j}\right|_{U_{i j}} \rightarrow \mathcal{F}_{i}\right|_{U_{i j}}$ be isomorphisms such that

$$
\varphi_{i i}=\mathrm{id}, \varphi_{i j} \varphi_{j i}=\mathrm{id}, \varphi_{i j} \varphi_{j k} \varphi_{k i}=\mathrm{id} \quad \text { for all } i, j, k \in I
$$

Then there exist a sheaf $\mathcal{F} \in \operatorname{Sh}(X)$ and isomorphisms $\varphi_{i}:\left.\mathcal{F}\right|_{U_{i}} \rightarrow \mathcal{F}_{i}$ such that $\left.\varphi_{i j} \circ \varphi_{j}\right|_{U_{i j}}=\left.\varphi_{i}\right|_{U_{i j}}$. Moreover, $\mathcal{F}, \varphi_{i}$ are unique up to canonical isomorphism.
Proof. Let $J=I^{2}$; on this set, introduce the following preorder:

$$
(i, j) \leqslant(k, \ell) \quad: \Leftrightarrow \quad(i, j)=(k, \ell) \text { or }(i, j)=(\ell, k)
$$

For $(m, n) \in J$, consider the open inclusion $j_{m n}: U_{m n} \rightarrow X$, and define $\mathcal{F}_{m n}=j_{m n *}\left(\left.\mathcal{F}_{m}\right|_{U_{m n}}\right)$.

Let $\psi_{a a}=\operatorname{id}_{\mathcal{F}_{a}}$ for $a \in J$. For $a=(m, n), b=(n, m)$, define

$$
\begin{aligned}
\psi_{b a} \in \operatorname{Hom}\left(\mathcal{F}_{m n}, \mathcal{F}_{n m}\right) & =\operatorname{Hom}\left(j_{m n}^{-1} j_{m n *}\left(\left.\mathcal{F}_{m}\right|_{U_{m n}}\right),\left.\mathcal{F}_{n}\right|_{U_{m n}}\right) \\
& =\operatorname{Hom}\left(\left.\mathcal{F}_{m}\right|_{U_{m n}},\left.\mathcal{F}_{n}\right|_{U_{m n}}\right)
\end{aligned}
$$

to be $\varphi_{n m}$. If $a=b$, then this is consistent with the previous definition, because $\varphi_{m m}=\mathrm{id}$.

By the assumption, $\left(\mathcal{F}_{a}, \psi_{a b}\right)$ is a projective system of sheaves on $X$, so the projective limit $\mathcal{F}$ exists by Proposition 3.10. We obtain morphisms $\psi_{a}: \mathcal{F} \rightarrow \mathcal{F}_{a}$ such that $\psi_{a b} \circ \psi_{b}=\psi_{a}$ for any $b \leqslant a$.

Let $a=(m, n), b=(n, m)$. Then $\left.\mathcal{F}_{a}\right|_{U_{m n}}=\left.\mathcal{F}_{m}\right|_{U_{m n}}$ and $\psi_{b a}=\varphi_{n m}$. Since $j_{m n}^{-1}$ commutes with projective limits, there exists, as indicated by the dotted line, a morphism $\left.\left.\mathcal{F}_{m}\right|_{U_{m n}} \rightarrow \mathcal{F}\right|_{U_{m n}}$ making the following diagram commutative:


In other words, $\left.\psi_{a}\right|_{U_{m n}}:\left.\left.\mathcal{F}\right|_{U_{m n}} \rightarrow \mathcal{F}_{m}\right|_{U_{m n}}$ is an isomorphism (we have constructed an inverse).

Since $\mathcal{F}_{m}$ is a sheaf, it is the projective limit of the direct images of $\left.\mathcal{F}_{m}\right|_{U_{m n}}$ under the open inclusions $U_{m n} \rightarrow U_{m}$, for $m$ fixed and $n$ variable. A similar remark applies to $\left.\mathcal{F}\right|_{U_{m}}$. Thus, we may construct $\varphi_{m}:\left.\mathcal{F}\right|_{U_{m}} \rightarrow \mathcal{F}_{m}$ as the projective limit of $\left.\psi_{a}\right|_{U_{m n}}$, for $a=(m, n)$. Then $\varphi_{m}$ is an isomorphism, and by definition,

$$
\left.\varphi_{m n} \circ \varphi_{m}\right|_{U_{m n}}=\left.\varphi_{m n} \circ \psi_{a}\right|_{U_{m n}}=\left.\psi_{b}\right|_{U_{m n}}=\left.\varphi_{n}\right|_{U_{m n}}
$$

This shows existence in the sense of our assertion, and uniqueness can be similarly derived from the uniqueness of projective limits.

## 4. SUPERMANIFOLDS

### 4.1. Basics.

Definition 4.1. A super-ringed space is a pair $X=(X, \mathcal{F})$ where $X_{0}$ is a topological space and $\mathcal{F}$ is a sheaf of (unital, associative) $\mathbb{R}$-superalgebras.

Here, an $\mathbb{R}$-superalgebra is a real unital associative algebra $A$ with a $\mathbb{Z} / 2 \mathbb{Z}$-grading $A=A_{0} \oplus A_{1}$ such that $A_{i} \cdot A_{j} \subset A_{i+j(\bmod 2)}$. By definition, morphisms of superalgebras are unital and preserve the grading.

A morphism of ringed spaces $X=\left(X_{0}, \mathcal{F}\right) \rightarrow Y=\left(Y_{0}, \mathcal{G}\right)$ is a pair $\left(f, f^{*}\right)$ where $f: X_{0} \rightarrow Y_{0}$ is a continuous map and $f^{*}: \mathcal{G} \rightarrow f_{*} \mathcal{F}$ (or, equivalently, $f^{-1} \mathcal{G} \rightarrow \mathcal{G}$ is a morphism of sheaves (of $\mathbb{R}$-superalgebras). One obtains the category SRSp.

Whenever $X=\left(X_{0}, \mathcal{F}\right)$ is a super-ringed space and $U \subset X_{0}$ is open, let $\left.X\right|_{U}=\left(U,\left.\mathcal{F}\right|_{U}\right)$. Such a super-ringed space is called an open subspace of $X$. Given an open subspace, there is an inclusion morphism $j_{U}=j_{U}^{X}:\left.X\right|_{U} \rightarrow X$, defined by $j_{U}=\left(j_{U, 0}, j_{U}^{*}\right)$ where $j_{U, 0}$ is the inclusion of $U$ in $X_{0}$, and $j_{U}^{*}: j_{U, 0}^{-1} \mathcal{F}=\left.\left.\mathcal{F}\right|_{U} \rightarrow \mathcal{F}\right|_{U}$ is the identity. A morphism $\varphi: X \rightarrow Y$ is called an open embedding if there are an open subset $U \subset Y_{0}$ and an isomorphism $\tilde{\varphi}:\left.X \rightarrow Y\right|_{U}$ such that $\varphi=j_{U} \circ \tilde{\varphi}$ where $j_{U}=j_{U}^{Y}$.

Definition 4.2. A supermanifold is a super-ringed space $X=\left(X_{0}, \mathcal{O}_{X}\right)$ over such that:
(1). $X_{0}$ is Hausdorff and paracompact, and
(2). for any $x \in X_{0}$, there exist an open neighbourhood $U$ of $x$ and an open embedding of $\left.X\right|_{U}$ into $\mathbb{R}^{p \mid q}=\left(\mathbb{R}^{p}, \mathcal{C}_{\mathbb{R}^{p}}^{\infty} \otimes \bigwedge\left(\mathbb{R}^{q}\right)^{*}\right)$ for some $p, q$.

In this case, we denote $\operatorname{dim}_{x} X=p \mid q$. If $p \mid q$ can be chosen independent of $x$, then we say that $X$ is of pure dimension or that $X$ has dimension $\operatorname{dim} X=p \mid q$. An open embedding $\varphi:\left.X\right|_{U} \rightarrow \mathbb{R}^{p \mid q}$ is called a local chart. Given a local chart $\varphi$, the tuple

$$
(y, \eta)=\left(y_{1}, \ldots, y_{p}, \eta_{1}, \ldots, \eta_{q}\right)=\left(\varphi^{*}\left(x_{1}\right), \ldots, \varphi^{*}\left(x_{q}\right), \varphi^{*}\left(\xi_{1}\right), \ldots, \varphi^{*}\left(\xi_{q}\right)\right)
$$

where $x_{j}$ are the coordinate projections on $\mathbb{R}^{p}$, and $\xi_{j}$ is the canonical basis of $\left(\mathbb{R}^{q}\right)^{*}$, is called a system of local coordinates. The open subspaces of $\mathbb{R}^{p \mid q}$ are called superdomains.

The category of SMan of supermanifolds is obtained as the full subcategory of SRSp consisting of supermanifolds.

Notation 4.3. If $X$ is a supermanifold, denote $\mathcal{N}_{X}$ the presheaf $U \mapsto \mathcal{N}_{X}(U)$ where $\mathcal{N}_{X}(U)$ is the set of nilpotent elements of $\mathcal{O}_{X}(U)$. If $X$ is of pure dimension (or if the odd part of $\operatorname{dim}_{x} X$ is bounded), then $\mathcal{N}_{X}$ is a sheaf.

Proposition 4.4. Let $X_{0}$ be a paracompact Hausdorff space. Assume given an open cover $\left(U_{j}\right)$, supermanifolds $X_{j}$ whose underlying topological space is $U_{j}$, and isomorphisms $\varphi_{i j}:\left.\left.X_{i}\right|_{U_{i j}} \rightarrow X_{j}\right|_{U_{i j}}$ such that over $U_{i j k}$,

$$
\varphi_{i j} \varphi_{j k}=\varphi_{i k} \quad \text { for all } i, j, k
$$

There there exists a supermanifold $X$ whose underlying topological space is $X_{0}$, together with isomorphisms $\varphi_{i}:\left.X_{i} \rightarrow X\right|_{U_{i}}$ such that $\varphi_{i} \circ \varphi_{i j}=\varphi_{j}$; these data are unique up to canonical isomorphism.

In fact, for any morphisms $\psi_{j}: X_{j} \rightarrow Y$ such that $\left.\psi_{i}\right|_{U_{i j}} \circ \varphi_{i j}=\left.\psi_{j}\right|_{U_{i j}}$, there is a unique morphism $\psi: X \rightarrow Y$ such that $\left.\psi\right|_{U_{i}} \circ \varphi_{i}=\psi_{i}$.

Proof. This follows directly from Proposition 3.18; note that the inductive system on the level of ringed spaces gets translated into a projective system on the level of sheaves.

Notation 4.5. The supermanifold introduced in Proposition 4.4 is called the supermanifold obtained by gluing the $X_{j}$ along the $\varphi_{i j}$.

### 4.2. The maximal ideals $\mathfrak{m}_{x}$.

Definition 4.6. Let $R$ be a unital ring. If $R$ possesses a unique maximal left ideal, then $R$ is called a local ring. In this case, the unique maximal ideal $\mathfrak{m}$ is always an ideal, and $R / \mathfrak{m}$ is a skew-field.
4.7. Fix a supermanifold $X$, and a point $x \in X_{0}$. Let $\mathfrak{m}_{X, x} \subset \mathcal{O}_{X, x}$ be the subset of the stalk which consists of non-invertible elements; in particular, one has $\mathcal{N}_{X, x} \subset \mathfrak{m}_{X, x}$. Let $p \mid q=\operatorname{dim}_{x} X$; the stalk is entirely determined by local data, so we may assume that $X$ is a open subspace of $\mathbb{R}^{p \mid q}$, that $X_{0}$ contains 0 , and that $x=0$.

Define

$$
\mathcal{O}_{0}^{p \mid q}:=\mathcal{O}_{\mathbb{R}^{p \mid q}, 0}, \mathcal{C}_{0}^{p}=\mathcal{C}_{\mathbb{R}^{p}, 0}^{\infty}, \mathcal{N}_{0}^{p \mid q}:=\mathcal{N}_{\mathbb{R}^{p \mid q}, 0}, \mathfrak{m}^{p \mid q}=\mathfrak{m}_{\mathbb{R}^{p \mid q}, 0}
$$

and let $\mathfrak{m}^{p} \subset \mathcal{C}_{0}^{p}$ be the set of germs $f$ such that $f(x)=0$. Then $\mathfrak{m}_{0}$ consists exactly of the non-invertible elements of $\mathcal{C}_{0}^{p}$, since a smooth function which is non-zero at a point is locally invertible. in particular, $\mathcal{C}_{0}^{p}=\mathbb{R} \oplus \mathfrak{m}^{p}$.

We have $\mathcal{O}_{0}=\mathcal{C}_{0}^{p} \oplus \mathcal{N}_{0}^{p \mid q}$. Let $f \in \mathcal{O}_{0}^{p \mid q}$, and decompose $f=g+h$ accordingly. If $f \notin \mathfrak{m}_{0} \oplus \mathcal{N}_{0}^{p \mid q}$, then $g \notin \mathfrak{m}_{0}$, and is thus invertible. We conclude that $f$ is invertible with inverse

$$
\sum_{k=0}^{\infty}\left(-g^{-1} h\right)^{k} g^{-1}
$$

where the sum is finite since $g^{-1} h$ is nilpotent. Hence, $\mathfrak{m}^{p \mid q} \subset \mathfrak{m}^{p} \oplus \mathcal{N}_{0}^{p \mid q}$, and the converse inclusion is obvious.

Thus, $\mathfrak{m}^{p \mid q}=\mathfrak{m}^{p} \oplus \mathcal{N}_{0}^{p \mid q}$; in particular, $\mathfrak{m}^{p \mid q}$ is an ideal, and $\mathcal{O}_{0}^{p \mid q}=\mathbb{R} \oplus \mathfrak{m}^{p \mid q}$. Observe also the following: Since $\left(\mathcal{N}_{0}^{p \mid q}\right)^{q+1}=0$, we have

$$
\begin{equation*}
\left(\mathfrak{m}^{p \mid q}\right)^{k}=\mathfrak{m}^{p}\left(\mathfrak{m}^{p \mid q}\right)^{k-1} \quad \text { for all } \quad k>q . \tag{4.1}
\end{equation*}
$$

Proposition 4.8. Let $X$ be a supermanifold, and $x \in X_{0}$. Then $\mathfrak{m}_{X, x}$ is the unique maximal ideal of $\mathcal{O}_{X, x}$; in particular, $\mathcal{O}_{X, x}$ is a local ring. Moreover, $\mathcal{O}_{X, x}=\mathbb{R} \oplus \mathfrak{m}_{X, x}$.

Proof. Observe that in any supercommutative superalgebra, any left ideal is an ideal. Thus, it suffices to show that $\mathfrak{m}_{x}$ is the unique maximal ideal.

By definition, $\mathfrak{m}_{x}$ contains any proper ideal of $\mathcal{O}_{X, x}$, and by the above considerations, it is an ideal. This proves the first statement, and the rest has already been checked.
4.9. For any open $U \subset X_{0}, f \in \mathcal{O}_{X}(U)$, and $x \in U$, we may define $f(x)$ to be the unique number $\lambda \in \mathbb{R}$ such that $f-\lambda \in \mathfrak{m}_{x}$. Then $\mathfrak{m}_{x}$ consists exactly of the germs $f$ such that $f(x)=0$.

We denote by $j_{0}^{*}(f)$ the function $x \mapsto f(x)$ on $U$. Using local coordinates, one sees that $j_{0}^{*}(f)$ is continuous, i.e. $j_{0}^{*}(f) \in \mathcal{C}_{X_{0}}(U)$. Let $\mathcal{O}_{X_{0}}$ be the subsheaf of $\mathcal{C}_{X_{0}}$ obtained in this way. Since ( $X_{0}, \mathcal{O}_{X_{0}}$ ) is locally isomorphic to ( $\mathbb{R}^{p}, \mathcal{C}_{\mathbb{R}^{p}}^{\infty}$ ), we have that ( $X_{0}, \mathcal{O}_{X_{0}}$ ) is a smooth manifold. By abuse of notation, we will denote it by $X_{0}$. The morphism $\left(\operatorname{id}_{X_{0}}, j_{0}^{*}\right): X_{0} \rightarrow X$ will be denoted by $j_{0}$. It is an open embedding of $X_{0}$ into $X$.

We say that $f \in \mathcal{O}_{X}(U)$ takes values in any given set $A \subset \mathbb{R}$ whenever the function $j_{0}^{*}(f)$ does. Hence, $f$ may take positive values, etc.

In the following proposition, $[-]_{0}$ denotes the functor that takes $X$ to $X_{0}$ and $\varphi$ to $\varphi_{0}$.
Proposition 4.10. Let $\varphi: X \rightarrow Y$ be a morphism of supermanifolds and $x \in X_{0}$. For $f \in \Gamma\left(\mathcal{O}_{Y}\right)$ and $x \in X_{0}, \varphi^{*}(f)(x)=f\left(\varphi_{0}(x)\right)$. In particular, we have that $\varphi^{*}\left(\mathfrak{m}_{Y, \varphi_{0}(x)}\right) \subset \mathfrak{m}_{X, x}$, and $j_{0}$ is a natural transformation $[-]_{0} \rightarrow \mathrm{id}$.
Proof. Let $f \in \mathcal{O}_{Y, \varphi_{0}(x)}$. Then $\varphi^{*}(f-\lambda)=\varphi^{*}(f)-\lambda$ for all $\lambda \in \mathbb{R}$. If $\lambda \neq f\left(\varphi_{0}(x)\right)$, then this quantity is invertible, since $\varphi^{*}(1)=1$. There exists a unique $\lambda$ such that $\varphi^{*}(f)-\lambda$ is not invertible, namely, $\varphi^{*}(f)(x)$. But this $\lambda$ must then be equal to $f\left(\varphi_{0}(x)\right)$. The other statements are immediate.
Proposition 4.11. Let $X$ be a supermanifold, $p \mid q=\operatorname{dim} X$. Let $f \in \Gamma\left(\mathcal{O}_{X}\right)$. If $j_{0}^{*}(f)=0$ and $f_{x} \in \mathfrak{m}_{x}^{q+1}$ for all $x \in X_{0}$, then $f=0$.
Proof. The question is local, so we may assume that $X$ is an open subspace of $\mathbb{R}^{p \mid q}$. By assumption, $f_{x} \in\left(\mathfrak{m}_{x}^{p \mid q}\right)^{q+1}=\mathfrak{m}_{x}^{p}\left(\mathfrak{m}_{x}^{p \mid q}\right)^{q}$ for all $x \in X_{0}$, where we have used (4.1). Let $(x, \xi)$ be the standard coordinates. We may uniquely decompose $f$ as $f=\sum_{\beta} f_{\beta} \xi^{\beta}$ where $f_{\beta} \in \mathcal{C}^{\infty}\left(X_{0}\right)$ and $\xi^{\beta}=\xi_{\beta_{1}} \cdots \xi_{\beta_{k}}$ for $\beta=\left\{\beta_{1}<\cdots<\beta_{k}\right\} \subset\{1, \ldots, q\}$. Then $f_{\beta}(x)=\left(f_{\beta}\right)_{x}(x)=0$ for all $x \in X_{0}$, whenever $|\beta|>0$. But $f_{\varnothing}=j_{0}^{*}(f)=0$ by assumption, so that $f=0$.

To bring the maximal ideal $\mathfrak{m}_{x}$ to use, we need some results from algebra.
Definition 4.12. Let $R$ be a unital ring. The Jacobson radical $J(R)$ of $R$ is the intersection of all maximal left ideals of $R$.

Lemma 4.13. Let $r \in R$. Then $r \in J=J(R)$ if and only if for all $s \in R$, 1 - sr has a left inverse.

Proof. Let $r \in J$ and assume, seeking a contradiction, that $R(1-s r) \subsetneq R$ for some $s \in R$. There exists a maximal left ideal $\mathfrak{m} \subset R$ containing $R(1-s r)$.

On the other hand, $J$ being a left ideal, we have $s r \in J \subset \mathfrak{m}$. But then $1=1-s r+s r \in \mathfrak{m}$, contradicting the maximality of $\mathfrak{m}$. Hence, $1-s r$ has a left inverse for any $s \in R$.

Conversely, let $r \notin J$. Then there exists a maximal left ideal $\mathfrak{m}$ of $R$ such that $r \notin \mathfrak{m}$. Since $\mathfrak{m}+R x$ is a left ideal containing $\mathfrak{m}$, it follows that $R=\mathfrak{m}+R x$, and there are $m \in \mathfrak{m}$ and $s \in R$ such that $1=m-s r$. Then $1-s r=m \in \mathfrak{m}$ cannot have a left inverse.

The following proposition is known as Nakayama's Lemma.
Proposition 4.14. Let $M \in{ }_{R} \operatorname{Mod}$ be finitely generated. If $J(R) \cdot M=M$, then $M=0$.
Proof. Let $J M=M$ where $J=J(R)$, and $X$ be a minimal set of non-zero generators for $M$. Assume that $X \neq \varnothing$, so that $X=\left\{m_{1}, \ldots, m_{n}\right\}$. There exists $r_{j} \in J$ such that $m_{1}=\sum_{j=1}^{n} r_{j} m_{j}$, so $\left(1-r_{1}\right)^{-1} m_{1}=\sum_{j=2}^{n} r_{j} m_{j}$. By Lemma 4.13, $1-r_{1}$ has a left inverse, so $m_{1}=\sum_{j=2}^{n}\left(1-r_{1}\right)^{-1} r_{j} m_{j}$, contradicting the minimality of $X$. Hence, $X=\varnothing$, and $M=0$.

Corollary 4.15. Let $R$ be a local ring and $M \in{ }_{R} \operatorname{Mod}$ be finitely generated. If the images of $x_{1}, \ldots, x_{n} \in M$ in $M / \mathfrak{m} M$ form a basis over the skew-field $R / \mathfrak{m}$, then $x_{1}, \ldots, x_{n}$ is a minimal set of generators for $M$.

Conversely, any minimal set of generators for $M$ arises in this way. For any such minimal set $x=\left(x_{1}, \ldots, x_{n}\right)$, and any $y=\left(y_{1}, \ldots, y_{n}\right), y_{i} \in M$, there exists a square matrix $A=\left(a_{i j}\right) \in R^{n \times n}$ such that $y=A x$, and $y$ is a set of generators if and only if some (any) such matrix $A \in \operatorname{GL}(n, R)$.

Proof. Since bases are minimal, the minimality is clear, once it has been established that $m_{1}, \ldots, m_{n}$ generate $M$. Let $N$ be the left submodule of $M$ generated by the $m_{j}$. By assumption $(N+\mathfrak{m} M) / \mathfrak{m} M=M / \mathfrak{m} M$, so $N+\mathfrak{m} M=M$. This implies that $\mathfrak{m} P=P$ for the module $P=M / N$. Since $\mathfrak{m}=J(R)$, Proposition 4.14 implies that $P=0$, so that $M=N$.

The converse statement is trivial, since a minimal set of generators of a vector space over a skew-field is automatically linearly independent.

Finally, let $x=\left(x_{1}, \ldots, x_{n}\right)$ be a minimal set of generators and let any tuple $y=\left(y_{1}, \ldots, y_{n}\right)$ of elements of $M$ be given. There are $a_{i j} \in R$ such that $y_{i}=\sum_{j=1}^{n} a_{i j} x_{j}$, so we may define $A=\left(a_{i j}\right)$. By the above, the row rank of $A(\bmod \mathfrak{m})$ is maximal if and only if $y$ is a set of generators, so the row determinant of $A(\bmod \mathfrak{m})$ is non-zero exactly in that case. It follows that $A$ is invertible if and only if $y$ is a set of generators.

In order to apply Nakayama's Lemma in the context of supermanifolds, we need to prove that $\mathfrak{m}_{x}$ is finitely generated. This is the content of the following proposition, whose classical counterpart is called Hadamard's Lemma.

Proposition 4.16. Let $X$ be a supermanifold, $x \in X_{0}$. Then $\mathfrak{m}_{x}$ is finitely generated; indeed, if $(y, \eta)$ is a system of local coordinates, then $\mathfrak{m}_{x}$ is generated by the germs of $y_{i}-y_{i}(x)$ and $\eta_{j}$. In particular, $\operatorname{dim}_{x} X=\operatorname{dim} \mathfrak{m}_{x} / \mathfrak{m}_{x}^{2}$.
Proof. Using the local coordinates, so we may assume that $X$ is an open subspace of $\mathbb{R}^{p \mid q}$ containing 0 , that $x=0$, and that $\left(y_{i}, \eta_{j}\right)=\left(x_{i}, \xi_{j}\right)$ are the standard coordinates. In the decomposition $\mathfrak{m}^{p \mid q}=\mathfrak{m}^{p} \oplus \mathcal{N}_{0}^{p \mid q}, \mathcal{N}_{0}^{p \mid q}$ is generated by $\xi_{j}$, so it is sufficient to see that $\mathfrak{m}^{p}$ is generated by $x_{i}$.

But if $f$ is a locally defined $\mathcal{C}^{\infty}$ function vanishing at 0 , then

$$
f(z)=\int_{0}^{1} f^{\prime}(t z) z d t=\sum_{j=1}^{n} z_{j} \cdot \int_{0}^{1} \partial_{j} f(t z) d t
$$

for $z$ in a convex neighbourhood of 0 . Since $z \mapsto \int_{0}^{1} \partial_{j} f(t z) d t$ are $\mathcal{C}^{\infty}$, this proves the assertion.

Clearly, $y_{i}-y_{i}(x), \eta_{j}$ form a minimal set of generators for $\mathfrak{m}_{x}$. By Nakayama's Lemma 4.14, $\operatorname{dim} \mathfrak{m}_{x} / \mathfrak{m}_{x}^{2}=p \mid q$. (Here, observe that the parity is preserved in the quotient.)
Corollary 4.17. Let $X$ be a supermanifold with coordinates $(y, \eta)$, and $f \in \Gamma\left(\mathcal{O}_{X}\right)$. For any $k \geqslant 0$, there exists a polynomial $p \in \mathbb{R}\left[u_{1}, \ldots, u_{p+q}\right]$ of degree $\leqslant k$ such that $(f-p(y-y(x), \eta))_{x} \in \mathfrak{m}_{x}^{k+1}$.
Proof. W.l.o.g. $y_{j}(x)=0$ for all $j$. We prove for all $k$ : If $f_{x} \in \mathfrak{m}_{x}^{k}$, then there exists $p$ such that $(f-p(y, \eta))_{x} \in \mathfrak{m}_{x}^{k+1}$. For $k=0$, we may take $p=f(x)$.

Assume that $k \geqslant 1$. By Nakayama's Lemma 4.14, the cosets of the monomials

$$
y^{\alpha} \eta^{\beta}=y_{1}^{\alpha_{1}} \cdots y_{p}^{\alpha_{p}} \eta_{1}^{\beta_{1}} \cdots \eta_{q}^{\beta_{q}}
$$

where $\alpha_{1}+\cdots+\alpha_{p}+\beta_{1}+\cdots+\beta_{q}=k, \alpha_{j} \in \mathbb{N}, \beta_{j} \in\{0,1\}$, form a basis of $\mathfrak{m}_{x}^{k} / \mathfrak{m}_{x}^{k+1}$. Thus, there is a linear combination $p(y, \eta)$ of such monomials, such that $(f-p(y, \eta))_{x} \in \mathfrak{m}_{x}^{k+1}$.
4.3. Morphisms, linear supermanifolds and products. The following basic fact justifies the definition of supermanifolds as ringed spaces: "Superfunctions", i.e. global sections of the structure sheaf, are exactly the same thing as morphisms $X \rightarrow \mathbb{R}^{1 \mid 1}$. In fact, we will want to have a more general statement, independent of dimension and the choice of particular coordinates. For this purpose, the following concept proves useful.
Definition 4.18. Let $V=V_{0} \oplus V_{1}$ be a super-vector space. We associate with $V$ the supermanifold given by $\left(V_{0}, \mathcal{C}_{V_{0}}^{\infty} \otimes \bigwedge\left(V_{1}\right)^{*}\right)$. We call this the linear supermanifold associated with $V$, and denote it by the same letter. Observe that $V^{*} \subset \Gamma\left(\mathcal{O}_{V}\right)$.

Given super vector spaces $V$ and $W$, we define a super vector space $V \otimes W$ by taking the grading $(V \otimes W)_{i}=\bigoplus_{k+\ell \equiv i(\bmod 2)} V_{k} \otimes W_{\ell}$. Similarly, we define a super-vector space $\underline{\operatorname{Hom}}(V, W)$ by taking the grading $\operatorname{Hom}(V, W)_{i}=$ $\bigoplus_{k+\ell \equiv i(\bmod 2)} \operatorname{Hom}\left(V_{k}, W_{\ell}\right)$. The even part of $\underline{\operatorname{Hom}}(V, W)$ is also denoted by $\operatorname{Hom}(V, W)$.

Lemma 4.19. Let $V$, $W$ be f.d. super vector spaces, and $A \in \operatorname{Hom}(V, W)$. Then there is a unique morphism $\varphi=\varphi_{A}: V \rightarrow W$ of the associated linear supermanifolds such that $\varphi^{*}(\mu)=\mu \circ A$ for all $\mu \in V^{*}$ and $\varphi^{*}(f)=f \circ A$ for all $f \in \mathcal{C}^{\infty}(U), U \subset W_{0}$ open.

Thus, the correspondence which sends f.d. super vector spaces to their associated linear supermanifolds is a functor.

Proof. Define $\varphi_{A, 0}(x)=A x$ for all $x \in V_{0}$. Let $\xi_{1}, \ldots, \xi_{q}$ be a basis of $W_{1}^{*}$. Any $f \in \mathcal{O}_{W}(U)$ may be decomposed uniquely as $f=\sum_{\beta} f_{\beta} \xi^{\beta}$ where the sum
is over $\beta=\left\{\beta_{1}<\cdots<\beta_{k}\right\} \subset\{1, \ldots, q\}, f_{\beta} \in \mathcal{C}^{\infty}(U)$, and $\xi^{\beta}=\xi_{\beta_{1}} \cdots \xi_{\beta_{k}}$. Define $\varphi_{A}^{*}(f)=\sum_{\beta}\left(f_{\beta} \circ A\right) \cdot\left(\xi_{\beta_{1}} \circ A\right) \cdots\left(\xi_{\beta_{q}} \circ A\right)$.

It is clear that $\varphi_{A}: V \rightarrow W$ is indeed a well-defined morphism. Moreover, $\varphi_{A}$ satisfies the condition stated in the assertion, and it is also trivial that this determines $\varphi_{A}$ uniquely. (In particular, $\varphi_{A}$ does not depend on the choice of a basis of $W_{1}^{*}$.)

Remark 4.20. A simple consequence of Lemma 4.19 is the following: For any linear supermanifold $V$, there is an algebra morphism $\mathcal{C}^{\infty}\left(V_{0}\right) \rightarrow \Gamma\left(\mathcal{O}_{V}\right)$, corresponding to the canonical morphism $V \rightarrow V_{0}$.

In particular, any smooth or analytic fucntion on $V_{0, \alpha}$ may be considered in a canonical way as a superfunction on $V$. As we shall see, this carries over to supermanifolds, however, in a non-canonical way.

Lemma 4.21. Let $V$ be a linear supermanifold of dimension $p \mid q$. Any basis of the underlying super vector space determines a system of local coordinates. In particular, there exist systems of local coordinates contained in $V^{*}$.

Proof. This is left to the reader.
Theorem 4.22. Let $X$ be a supermanifold and $V$ a linear supermanifold. There is a bijection

$$
V(X) \xrightarrow{\cong} \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right)=\left(\Gamma\left(\mathcal{O}_{X}\right) \otimes V\right)_{0}
$$

which is natural in $X$. Under this map,

$$
\operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right) \quad \text { and } \quad \operatorname{Hom}\left(V_{1}^{*}, \Gamma\left(\mathcal{O}_{X}\right)_{1}\right)
$$

correspond to the $X$-points of $V_{0}$ and $V_{1}$, respectively.
Remark 4.23. In particular, an element of $\Gamma\left(\mathcal{O}_{X}\right)$ represents the same data as a morphism $X \rightarrow \mathbb{R}^{111}$. The naturality of this identification amounts to the following: If $f \in \Gamma\left(\mathcal{O}_{X}\right)$ and $\varphi: Y \rightarrow X$ is a morphism, then the morphism $Y \rightarrow \mathbb{R}^{1 \mid 1}$ corresponding to $\varphi^{*}(f)$ is $f \circ \varphi$, where $f$ is considered as a morphism $X \rightarrow \mathbb{R}^{1 \mid 1}$.

Proof of Theorem 4.22. Define

$$
\phi: V(X) \rightarrow \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right): \varphi \mapsto \phi(\varphi), \phi(\varphi)(\mu)=\varphi^{*}(\mu) .
$$

By Proposition 4.10,

$$
\phi(\varphi)(\mu)(x)=\varphi^{*}(\mu)(x)=\mu\left(\varphi_{0}(x)\right) \quad \text { for all } x \in X_{0}, \mu \in V_{0}^{*}
$$

This already determines $\varphi_{0}$. It is also readily seen by this definition that $\phi=\phi_{X}$ is natural in $X$. The statements about the $X$-points of $V_{0}$ and $V_{1}$ are special cases of naturality.

The map $\phi$ is injective. Indeed, let $\phi\left(\varphi_{1}\right)=\phi\left(\varphi_{2}\right)$, so $\varphi_{1}^{*}(\mu)=\varphi_{2}^{*}(\mu)$ for all $\mu \in V^{*}$. The morphisms $\varphi_{1}, \varphi_{2}$ have the same underlying map $\varphi_{0}$.

Let $f \in \mathcal{O}_{V}(U)$ and $x \in \varphi_{0}^{-1}(U)$. We may choose local coordinates $(y, \eta)$ in $V^{*}$, by Lemma 4.21. By Corollary 4.17, there exists a polynomial $t$ in $(y, \eta)$ such that $(f-t)_{\varphi_{0}(x)} \in \mathfrak{m}_{\varphi_{0}(x)}^{q+1}$ where $p \mid q=\operatorname{dim}_{x} X$. Since $\varphi_{1}^{*}$ and $\varphi_{2}^{*}$ coincide on $t$ by assumption, and by Proposition 4.10,

$$
\left(\varphi_{1}^{*}(f)-\varphi_{2}^{*}(f)\right)_{x}=\varphi_{1}^{*}(f-t)_{x}-\varphi_{2}^{*}(f-t)_{x} \in \mathfrak{m}_{X, x}^{q+1}
$$

Since $j_{0}^{*}\left(\varphi_{1}^{*}(f)-\varphi_{2}^{*}(f)\right)=j_{0}^{*}(f) \circ \varphi_{0}-j_{0}^{*}(f) \circ \varphi_{0}=0$, and because $x$ was arbitrary, Proposition 4.11 shows that $\varphi_{1}^{*}(f)=\varphi_{2}^{*}(f)$.

To see that $\phi$ is surjective, let us first assume that $X$ is a supermanifold. We will define an inverse map $\psi: \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right) \rightarrow V(X)$. To that end, fix $f \in \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right)_{0}$. We wish to define $\varphi=\psi(f)$. First, define $\varphi_{0}: X_{0} \rightarrow V_{0}$ by the equation

$$
\mu\left(\varphi_{0}(x)\right)=f(\mu)(x)=j_{0}^{*}(f(\mu))(x) \quad \text { for all } \mu \in V_{0}^{*} .
$$

Since $j_{0}^{*}(f(\mu))$ is smooth for any $\mu$, so is $\varphi_{0}$.
Next, we define $\varphi^{*}: \mathcal{O}_{V} \rightarrow \varphi_{0, *} \mathcal{O}_{X}$. Referring to the coordinates $(y, \eta)$, it will be sufficient to define $\varphi^{*}$ on $\mathcal{C}^{\infty}(U)$, where $U \subset V_{0}$ is open, since $\eta_{j}$, generate $\mathcal{O}_{V}(U)$ as a $\mathcal{C}^{\infty}(U)$-algebra. Thus, let $h \in \mathcal{C}^{\infty}(U)$. Define $\varepsilon_{j}=f\left(y_{j}\right)-y_{j} \circ \varphi_{0} \in \mathcal{O}_{X}\left(\varphi_{0}^{-1}(U)\right)$ and

$$
\varphi^{*}(h)=\sum_{\alpha \in \mathbb{N}^{m}} \frac{1}{\alpha!} \cdot\left(h^{(\alpha)} \circ \varphi_{0}\right) \cdot \varepsilon^{\alpha} \quad \text { where } \quad \varepsilon^{\alpha}=\varepsilon_{1}^{\alpha_{1}} \ldots \varepsilon_{m}^{\alpha_{m}}
$$

 This makes sense because $\varepsilon$ is nilpotent, and because $X$ is a superdomain.

To see that $\varphi^{*}$ is an algebra morphism, observe that $\varphi^{*}(1)=1$, and compute

$$
\varphi^{*}\left(h_{1} h_{2}\right)=\sum_{\gamma} \sum_{\alpha+\beta=\gamma} \frac{1}{\alpha!\beta!} \cdot\left(h_{1}^{(\alpha)} \circ \varphi_{0}\right) \cdot\left(h_{2}^{(\beta)} \circ \varphi_{0}\right) \cdot n^{\gamma}=\varphi^{*}\left(h_{1}\right) \cdot \varphi^{*}\left(h_{2}\right) .
$$

Given $f \in \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right)_{\alpha}$, let $\tilde{f}(\mu)=\phi(\psi(f))(\mu)=\psi(f)^{*}(\mu)$. Since $y_{j}^{(\alpha)}=\delta_{\alpha 0} y_{j}+\delta_{\alpha, e_{j}}$, we have $\tilde{f}\left(y_{j}\right)=f\left(y_{j}\right)$. This implies $\tilde{f}=f$, so $\phi \circ \psi=\mathrm{id}$, and in particular, $\phi$ is surjective in the case of a superdomain.

Back in the general case, write $\phi=\phi_{X}$ and choose some covering of $X$ by open subspaces $X_{i}=\left.X\right|_{U_{i}}$ which are isomorphic to superdomains. Let $f \in \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right)$. Define $f_{i} \in \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X_{i}}\right)\right)$ by $f_{i}(\mu)=\left.f(\mu)\right|_{U_{i}}$. There exist $\varphi_{i} \in V\left(X_{i}\right)$ such that $\phi_{X_{i}}\left(\varphi_{i}\right)=f_{i}$. By Proposition 4.4, there exists $\varphi \in V(X)$ such that $\left.\varphi\right|_{U_{i}}=f \circ j_{U_{i}}=\varphi_{i}$. Then

$$
\left.\phi(\varphi)(\mu)\right|_{U_{i}}=j_{U_{i}}^{*} \varphi^{*}(\mu)=\varphi_{i}^{*}(\mu)=\phi_{X_{i}}\left(\varphi_{i}\right)(\mu)=f_{i}(\mu)=\left.f(\mu)\right|_{U_{i}}
$$

for all $\mu \in V^{*}$, and all $i$. By the sheaf axioms, this proves equality, and thus that $\phi$ is surjective.

Corollary 4.24. Let $V$ and $W$ be linear supermanifolds. Then the linear supermanifold corresponding to the super-vector space $V \times W$ is the product of $V$ and $W$ in the category SMan.
Proof. Let $U=V \times W$ as super vector spaces. For any $X$,

$$
\begin{aligned}
U(X) & \cong \operatorname{Hom}\left(U^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right) \\
& =\operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right) \times \operatorname{Hom}\left(W^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right) \cong V(X) \times W(X) .
\end{aligned}
$$

By the naturality in Theorem 4.22, this bijection is natural. Therefore, by Corollary 2.25 , we conclude that $U=V \times W$ as supermanifolds.

In fact, the previous to results carry over to superdomains. This follows from the following simple observation.

Lemma 4.25. Let $X, Y$ be supermanifolds, $U$ an open subspace of $Y$, and $f: X \rightarrow Y$ a morphism. Then $f$ factors through the inclusion $U \rightarrow Y$ if and only if $f_{0}$ takes its values in $U_{0}$.

Proof. This is a trivial consequence of the definition of morphisms.
Corollary 4.26. Let $X$ be a supermanifold and $U$ an open subspace of $\mathbb{R}^{p \mid q}$. There is a natural bijection

$$
U(X) \xrightarrow{\cong}\left\{(y, \eta) \in \Gamma\left(\mathcal{O}_{X}\right)_{0}^{p} \times \Gamma\left(\mathcal{O}_{X}\right)^{q} \mid \forall x \in X_{0}: y(x) \in U_{0}\right\} .
$$

Corollary 4.27. Let $U_{j}$ be an open subspace of $\mathbb{R}^{p_{j} \mid q_{j}}, j=1,2$. The open subspace of $\mathbb{R}^{p_{1}+p_{2} \mid q_{1}+q_{2}}$ corresponding to the open subset $U_{1,0} \times U_{2,0} \subset \mathbb{R}^{p_{1}+p_{2}}$ is the product of $U_{1}$ and $U_{2}$ in the category SMan.

Corollary 4.28. Finite direct products exist in SMan.
Proof. A terminal object is given by $(*, \mathbb{R})$ where $*$ is any singleton set and $\mathbb{R}$ is the constant sheaf on $*$. (An algebra morphism $\mathbb{R} \rightarrow \Gamma\left(\mathcal{O}_{X}\right)$ is fixed by its value on 1 , so there is only one.) This covers the case of empty products.

Binary, and thus, arbitrary finite products, exist in view of Corollary 4.27 and Proposition 4.4.

Example 4.29. Let $\left(x, \xi_{1}, \xi_{2}\right)$ be the standard coordinate system of $\mathbb{R}^{1 \mid 2}$. There is an automorphism of $\mathbb{R}^{1 \mid 2}$, given by $\left(x, \xi_{1}, \xi_{2}\right) \mapsto\left(x+\xi_{1} \xi_{2}, \xi_{1}, \xi_{2}\right)$. This example shows that morphisms need not respect the $\mathbb{Z}$-grading on $\mathcal{O}_{\mathbb{R}^{p \mid q}}$.

The existence of finite products allows for the following definition.
Definition 4.30. A Lie supergroup is a group object in SMan.
Exercise 4.31. One defines the complexification functor $[-]_{\mathbb{C}}$ from real super-ringed spaces to complex super-ringed spaces as follows: On objects, $(X, \mathcal{F})_{\mathbb{C}}=\left(X, \mathcal{F} \otimes_{\mathbb{R}} \mathbb{C}\right) ;$ on morphisms $\left(f, f^{*}\right)_{\mathbb{C}}=\left(f, f^{*} \otimes_{\mathbb{R}} \mathrm{id}_{\mathbb{C}}\right)$.
(1). Show that $[-]_{\mathbb{C}}$ induces a fully faithful functor from real manifolds.
(2). Show that this functor, when considered on real smooth supermanifolds, is neither full nor faithful.

Exercise 4.32. Show that the definition of manifolds in terms of ringed spaces is equivalent to the definition in terms of atlases, in fact, that these are isomorphic categories. Formulate a definition (equivalent to the ringed space definition) of supermanifolds and their morphisms in terms of atlases.

Exercise 4.33. A superpoint is a supermanifold whose underlying manifold is a point. Given supermanifolds $X$ and $Y$, show that morphisms $f: X \rightarrow Y$ are uniquely determined by $f(\lambda): X(\lambda) \rightarrow Y(\lambda)$, where $\lambda$ runs though all superpoints.

Exercise 4.34. Show that there are inequivalent Lie supergroup structures on $\mathbb{R}^{1 \mid 1}$. Find all non-isomorphic Lie supergroup structures on $\mathbb{R}^{1 \mid 1}$.

### 4.4. Weil superalgebras and inner homs.

4.35. Let $R$ be a commutative ring and let $\operatorname{Spec} R$ (the spectrum of $R$ ) denote the set of prime ideals of $R$ where a proper ideal $\mathfrak{p} \subsetneq R$ is prime if $a b \in \mathfrak{p}$ implies that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, for all $a, b \in R$. For any $f \in R$, define the fundamental open set $U_{f}$ as

$$
U_{f}=\{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}
$$

These sets generate a topology on $\operatorname{Spec} R$, called the Zariski topology.
Let $A$ be a supercommutative $\mathbb{R}$-superalgebra. With the Zariski topology, Spec $A_{0}$ will be the underlying topological space of $\operatorname{Spec} A$; its structure sheaf of $\operatorname{Spec} A$ is defined by localisation. Let us recall this construction.

Construction 4.36. Let $R$ be a commutative ring and $S \subset R$. The set $S$ is called multiplicative if $1 \in S$ and $S \cdot S \subset S$. Let $S$ be multiplicative.

Let $M$ be an $R$-module. Define an equivalence relation $\sim$ on $S \times M$ by

$$
(s, m) \sim(t, n) \quad: \Leftrightarrow \quad \exists r \in S: r(t n-s m)=0
$$

Let $S^{-1} M$ the quotient of $S \times M$ by this equivalence relation; it is called the localisation of $M$ at $S$. The element of $S^{-1} M$ represented by $(s, m)$ is denoted $s^{-1} m$. Then $S^{-1} M$ is an $R$-module via

$$
r \cdot\left(s^{-1} m\right)=s^{-1}(r m), s^{-1} m+t^{-1} n=(s t)^{-1}(t m+s n)
$$

The $R$-module $S^{-1} M$ comes with the canonical $R$-module morphism $M \rightarrow S^{-1} M: m \mapsto 1^{-1} m$. These data enjoy the following universal property: Given any $R$-module morphism $\phi: M \rightarrow N$ such that any $s \in S$ acts on $N$ by an automorphism, there is a unique factorisation of $\phi$ through an $R$-module morphism $S^{-1} M \rightarrow N$.

If $M=R$, then $S^{-1} R=S^{-1} M$ has, in addition, a ring structure, given by

$$
s^{-1} m \cdot t^{-1} n=(s t)^{-1}(m n) ;
$$

this is the localised ring of $R$ at $S$. It is straightforward to see that $S^{-1} M$ is an $S^{-1} R$-module, for any $R$-module $M$.

If $S=R \backslash \mathfrak{p}$ for a prime ideal $\mathfrak{p}$, one denotes $M_{\mathfrak{p}}=S^{-1} M, R_{\mathfrak{p}}=S^{-1} R$; similarly, when $S=\left\{f^{k} \mid k=0,1, \ldots\right\}$ where $f \in R$, one writes $M_{f}=S^{-1} M$, $R_{f}=S^{-1} R$.
Proposition 4.37. Let $A$ be a supercommutative $\mathbb{R}$-superalgebra. For $f \in A_{0}$, let $A_{f}$ be the localisation of the $A_{0}$-module $A$ at $f$. The correspondence $U_{f} \mapsto A_{f}$ extends uniquely (up to unique isomorphism) to a sheaf on the space $\operatorname{Spec} A_{0}$.

In the proof, we will need the following lemma.
Lemma 4.38. Let $R$ be a commutative ring. The topological space $\operatorname{Spec} R$ is quasi-compact, that is, any open cover has a finite subcover.
Proof. It is sufficient to prove this for a cover by fundamental open sets. Thus, let $\operatorname{Spec} R=\bigcup_{j} U_{f_{j}}$ where $f_{j} \in R$. Let $I$ be the ideal generated by the $f_{j}$; if $I \neq R$, then $I$ would be contained in a maximal ideal $\mathfrak{m}$. This cannot be the case, since maximal ideals are prime, and $I$ is not contained in a prime ideal, in view of our assumption. Thus, $R=I$ and $1=\sum_{j} r_{j} f_{j}$ for some $r_{j} \in R$, all but finitely many of which are $=0$.

Let $F$ be the set of all $f_{j}, r_{j} \neq 0$. If $\mathfrak{p}$ is a prime ideal, then $F \not \subset \mathfrak{p}$ since otherwise $1 \in \mathfrak{p}$. Thus, Spec $R=\bigcup_{f \in F} U_{f}$ and we have obtained the required finite subcover.

Proof of Proposition 4.37. In view of Proposition 3.18, it is sufficient to show that this correspondence defines sheaves on the fundamental open subsets of each $U_{f}$. Moreover, since $A_{0}$ is central in $A$ it is clear that same the definition as for the multiplication on $\left(A_{0}\right)_{f}$ endows $A_{f}$ with a supercommutative superalgebra structure.

Let $f \in A_{0}$. Observe that $\operatorname{Spec}\left(A_{0}\right)_{f}=U_{f}$. Indeed, any prime ideal in $\left(A_{0}\right)_{f}$ corresponds under the canonical map to a prime ideal $\mathfrak{p} \subset A_{0}$; for it to be proper in $\left(A_{0}\right)_{f}$, one need to have $f \notin \mathfrak{p}$, since $f$ is invertible in $\left(A_{0}\right)_{f}$. The converse is also true. Moreover, the fundamental open subset of $\operatorname{Spec}\left(A_{0}\right)_{f}$ corresponding to $g$ is $U_{f g}$ since $f g \notin \mathfrak{p}$ is equivalent to $g \notin \mathfrak{p}$, in view of the primality of $\mathfrak{p}$. Since $U_{f g}=U_{f} \cap U_{g}$, the identification of $\operatorname{Spec}\left(A_{0}\right)_{f}$ and $U_{f}$ is a homeomorphism.

A similar argument shows $\left(A_{f}\right)_{g}=A_{f g}$; this defines the restriction morphisms of the sought-for sheaf. Moreover, replacing $A$ by $A_{f}$ and $U_{f}$ by $\operatorname{Spec} A$, it will be sufficient to prove that on the basis of open subsets $U_{f}$, $U_{f} \mapsto A_{f}$ defines a sheaf. By Lemma 4.38, it will suffice to check the sheaf axioms for finite open covers.

Thus, let $g \in \operatorname{Spec} A_{0}=\bigcup_{j=1}^{n} U_{f_{j}}$ such that $\left.g\right|_{U_{f_{j}}}=0$ for all $j$. Thus, for all $j$, there is $m_{j}$ such that $f_{j}^{m_{j}} g=0$. Let $m=m_{1} \cdots m_{n}$, then $f_{j}^{m} g=0$ for all $j$. Let $I=\left(f_{1}, \ldots, f_{n}\right) \subset A_{0}$ and $I_{m}=\left(f_{1}^{m}, \ldots, f_{n}^{m}\right) \subset A_{0}$; then $I^{m \cdot n} \subset I_{m}$. But $I=A_{0}$ by assumption, so $I_{m}=A_{0}$. Since $r \cdot g=0$ for all $r \in I_{m}$ and we may take $r=1$, it follows that $g=0$.

Next, let $g_{j} \in U_{f_{j}}$ be given such that $\left.g_{i}\right|_{U_{f_{i}} f_{j}}=\left.g_{j}\right|_{U_{f_{i} f_{j}}}$. There is some $N>0$ such that

$$
\left(f_{i} f_{j}\right)^{N} g_{i}=\left(f_{i} f_{j}\right)^{N} g_{j} \text { for all } i, j
$$

Possibly enlarging $N$, we may assume $f_{j}^{N} g_{j} \in A$, and, as above, that $1=\sum_{j=1}^{n} r_{j} f_{j}^{N}$ for some $r_{j} \in A_{0}$. Thus, set $g=\sum_{j=1}^{n} r_{j} f_{j}^{N} g_{j}$. Then

$$
f_{i}^{N} g=\sum_{j=1}^{N} r_{j}\left(f_{i} f_{j}\right)^{N} g_{j}=\sum_{j=1}^{n} r_{j} f_{j}^{N} \cdot f_{i}^{N} g_{i}=f_{i}^{N} g_{i}
$$

so $\left.g\right|_{U_{f_{i}}}=g_{i}$, as required.
Definition 4.39. Let $\operatorname{Spec} A=\left(\operatorname{Spec} A_{0}, \mathcal{O}_{A}\right)$ where $\mathcal{O}_{A}$ is the sheaf defined in Proposition 4.37, considered as an object of the category $\operatorname{SRSp}_{\mathbb{R}}$. This super-ringed space is called the spectrum of $A$.

We will apply this construction in a very particular case.
Definition 4.40. A Weil superalgebra is a finite-dimensional $\mathbb{R}$-superalgebra $A$ with a graded nilpotent ideal $J_{A}$ such that $A=\mathbb{R} \oplus J_{A}$.

Any Weil superalgebra is a local ring, and $J_{A}$ is the maximal ideal; it consists exactly of the nilpotent elements of $A$.
Exercise 4.41. Let $A$ be a $\mathbb{R}$-superalgebra. Then $A$ is a Weil superalgebra if and only if $A \cong \mathbb{R}\left[x_{1}, \ldots, x_{k}, \xi_{1}, \ldots, \xi_{\ell}\right] / I$ for some graded ideal $I$ such that $I \supset\left(x_{1}, \ldots, x_{k}, \xi_{1}, \ldots, \xi_{\ell}\right)^{N}$, for some $N>0$.

Lemma 4.42. Let $A$ be a Weil superalgebra. Then $\operatorname{Spec} A_{0}$ is a point.
Proof. We may and will assume $A=A_{0}$. Then $A=\mathbb{R}\left[x_{1}, \ldots, x_{k}\right] / I$ for some graded ideal $I \supset\left(x_{1}, \ldots, x_{k}\right)^{N}$. Any prime ideal $\mathfrak{p} \subset A$ has the form $\mathfrak{q} / I$ where $\mathfrak{q} \subset \mathbb{R}\left[x_{1}, \ldots, x_{k}\right]$ is a prime ideal containing $I$. Then $x_{j}^{N} \in \mathfrak{q}$ for all $j$. Being a prime ideal, $\mathfrak{q}$ is radical, so $x_{j} \in \mathfrak{q}$. Then $\mathfrak{q}=\left(x_{1}, \ldots, x_{k}\right)$, and this determines $\mathfrak{p}=J_{A}$.

Remark 4.43. In fact, as follows from its proof, the conclusion of Proposition 4.10 holds for any super-ringed spaces $X, Y$ such that the stalks $\mathcal{O}=\mathcal{O}_{X, x}$ of the structure sheaves satisfy the following assumption: $\mathcal{O}$ is local, $\mathcal{O}=\mathbb{R} \oplus \mathfrak{m}$, and $\mathcal{O} \backslash \mathfrak{m}$ consists of invertible elements. For any such super-ringed space $X$, values of local sections of the structure sheaves can be defined as for supermanifolds. One obtains a subsheaf $\mathcal{O}_{X_{0}}$ of the sheaf of all $\mathbb{R}$-valued functions on the topological space $X_{0}$ and a canonical morphism $j_{0}: X_{0} \rightarrow X$ where we let $X_{0}=\left(X_{0}, \mathcal{O}_{X_{0}}\right)$ (by abuse of notation).

If, in addition, there exists some $q \gg 0$ such that any open subpace of $X$ satisfies the conclusion of Proposition 4.11, then by the proof of Theorem 4.22, for any linear supermanifold $V$, the canonical map $V(X) \rightarrow \operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{X}\right)\right)$ is injective. If, moreover, $X$ is covered by open subspaces $U$ such that $\mathcal{O}_{U}$ is an $\mathcal{O}_{U_{0}}$-algebra, then this map is also surjective.

We shall call $X$ locally determined, if these three properties are fulfilled.
Lemma 4.44. Let $X=\left(X_{0}, \mathcal{O}_{X}\right)$ be a super-ringed space and $A$ a Weil superalgebra. The product $X \times \operatorname{Spec} A$ exists in $\mathrm{SRSp}_{\mathbb{R}}$, and there exists a canonical right inverse $j_{A}: X \rightarrow X \times \operatorname{Spec} A$ to $p_{1}: X \times \operatorname{Spec} A \rightarrow X$, defining a natural transformation id $\rightarrow(-) \times \operatorname{Spec} A$. If $B$ is another Weil superalgebra, then $\operatorname{Spec}(A \otimes B)=\operatorname{Spec} A \times \operatorname{Spec} B$. If $X$ is a locally determined super-ringed space, then so is $X \times \operatorname{Spec} A$.
Proof. Let $Z=\left(X_{0}, \mathcal{O}_{Z}\right)$ where $\mathcal{O}_{Z}=\mathcal{O}_{X} \otimes A$, the tensor product of $\mathbb{R}$ superalgebras. For any sheaf $\mathcal{F}$ of supercommutative superalgebras on $X_{0}$, we have the isomorphism

$$
\operatorname{Hom}\left(\mathcal{O}_{X}, \mathcal{F}\right) \times \operatorname{Hom}(A, \mathcal{F}) \cong \operatorname{Hom}\left(\mathcal{O}_{X} \otimes A, \mathcal{F}\right)
$$

given by $\left(\varphi_{1}, \varphi_{2}\right) \mapsto \varphi_{1} \otimes \varphi_{2},\left(\varphi_{1} \otimes \varphi_{2}\right)(f \otimes a)=\varphi_{1}(f) \cdot \varphi_{2}(a)$. Here, Hom denotes morphisms of superalgebra sheaves. This proves that $Z=X \times \operatorname{Spec} A$.

The right inverse $j_{A}$ is defined by $j_{A, 0}=\operatorname{id}_{X}$ and by setting $j_{A}^{*}(f \otimes 1)=f$, $j_{A}^{*}(f \otimes a)=0$ for all $a \in J_{A}$. It is easy to check that this is natural.

The construction immediately shows that $\operatorname{Spec}(A \otimes B)=\operatorname{Spec} A \times \operatorname{Spec} B$.
Let $X$ be locally determined. It is straightforward that maximal ideal of $\mathcal{O}_{X, x} \otimes A$ is $\mathfrak{m}_{X, x} \otimes A \oplus \mathbb{R} \otimes J_{A}$, so indeed $\mathcal{O}_{X \times \operatorname{Spec} A, x}=\mathbb{R} \oplus \mathfrak{m}_{X \times \operatorname{Spec} A, x}$.

The elements of $\mathbb{R} \otimes J_{A}$ are nilpotent, so that $f \notin \mathfrak{m}_{X \times \operatorname{Spec} A, x}$ are certainly invertible. Moreover, $J_{A}^{N}=0$ for some $N \gg 0$, so that

$$
\mathfrak{m}_{X \times \operatorname{Spec} A, x}^{m+N}=\left(\mathfrak{m}_{X, x} \otimes A\right)^{m}\left(\mathfrak{m}_{X \times \operatorname{Spec} A, x}\right)^{N} \subset \mathfrak{m}_{X, x}^{m} \otimes A
$$

It follows easily that $X \times \operatorname{Spec} A$ is locally determined, since $\operatorname{Spec} A_{0}$ is a point, and $\mathcal{O}_{\text {Spec } A_{0}}=\mathbb{R}$. This proves the lemma.

Definition 4.45. Let $\mathcal{C}$ be a category and $X, Y \in \operatorname{Ob\mathcal {C}}$. Assume that $S \times X$ exists in $\mathcal{C}$ for any $S \in \mathrm{Ob} \mathcal{C}$. An object $Z$ is called inner hom from $X$ to $Y$
and denoted by $\underline{\operatorname{Hom}}(X, Y)$ if there is a natural isomorphism

$$
\operatorname{Hom}(X, Z)=\operatorname{Hom}(S, \underline{\operatorname{Hom}}(X, Y)) \cong \operatorname{Hom}(S \times X, Y),
$$

in other words, if $Z$ represents the functor $S \mapsto \operatorname{Hom}(S \times X, Y)$. If it exists, Hom $(X, Y)$ is unique up to canonical isomorphism.

We will use this concept in a slightly more general context: To define when an inner hom exists in supermanifolds, we will only require the products $S \times X$ to exist in the category of super-ringed spaces.
Remark 4.46. This definition mimics the customary identification of maps of sets $f: S \times X \rightarrow Y$ and $g: S \rightarrow \operatorname{Maps}(X, Y)$, given by $f(s, x)=g(s)(x)$.

The inner homs $\underline{\operatorname{Hom}}(\operatorname{Spec} A, X)$ will have the structure of a fibre bundle, so we introduce the relevant terminology.

Definition 4.47. Let $p: X \rightarrow S$ be a morphism of supermanifolds.
If $F$ is another supermanifold and $U$ an open subspace of $S$, then a local trivialisation of $p$ over $U$ with fibre $F$ is an open embedding $\tau: U \times F \rightarrow M$ such that $p \circ \tau=j_{U} \circ p_{1}$, i.e., the following diagram commutes:

where $p_{1}$ is the first projection. The open subspace $\left.X\right|_{p_{0}^{-1}\left(U_{0}\right)}$ of $X$ lying over $U$ is denoted by $\left.X\right|_{U}$; thus, $\tau$ induces an isomorphism $U \times\left. F \rightarrow X\right|_{U}$.

The set of local trivialisations of $p$ over $U$ is denoted by $\tau_{p}(U)$ or $\tau_{X}(U)$; similarly, the set of local trivialisations with fibre $F$ is denoted by $\tau_{p}^{F}(U)$ or $\tau_{X}^{F}(U)$. Then $\tau_{X}$ and $\tau_{X}^{F}$ are sheaves of sets. (Here, we identify open subspaces of $S$ with open subsets of $S_{0}$.)

A subsheaf $\mathcal{A}$ of $\tau_{X}$ resp. $\tau_{X}^{F}$ such that there exists an open cover $\mathcal{U}$ of $S$ such that $\mathcal{A}(U) \neq \varnothing$ for all $U \in \mathcal{U}$ is called an atlas of local trivialisations resp. of local trivialisations with fibre $F$. Then $p$ or, by a slur of language, $X$, is called
(1). an $S$-family (of supermanifolds) or a (relative) supermanifold over $S$, if $p$ possesses an atlas of local trivialisations, and
(2). a fibre bundle with fibre $F$, total space $X$, and base space $S$, if $p$ possesses an atlas of local trivialisations with fibre $F$.

Usually, the morphism $p=p_{X}$ will be understood in the notation, and one writes $X / S$ for $X$. We will adopt the same language for arbitrary morphisms $X \rightarrow S$ of super-ringed spaces, and call these (relative) super-ringed spaces over $S$. A relative morphism $X / S \rightarrow Y / S$ (of super-ringed spaces over $S$ ) is a morphism $f: X \rightarrow Y$ such that $p_{Y} \circ f=p_{X}$. More generally, if $X / S, Y / T$, then given $\varphi: S \rightarrow T$, we say that $\psi: X \rightarrow Y$ is over $\varphi$ if $\varphi \circ p_{X}=p_{Y} \circ \psi$.

One obtains a category of super-ringed spaces over $S$; the categories of supermanifolds over $S$, and of fibre bundles with fibre $F$ and base $S$ are defined as full subcategories.

As usual, there is a patching lemma.

Proposition 4.48. Let $S$ be a supermanifold, $\left(U_{i}\right)$ an open cover, and $F_{i}$ supermanifolds. Consider the trivial families $X_{i}=U_{i} \times F_{i}$ over $U_{i}$. Assume there are isomorphisms $\varphi_{i j}:\left.\left.X_{j}\right|_{U_{i j}} \rightarrow X_{i}\right|_{U_{i j}}$ over $U_{i j}$, such that

$$
\varphi_{i j} \circ \varphi_{j k}=\varphi_{i k},
$$

where the latter equation is understood over $U_{i j k}$.
Then there exist a supermanifold $X / S$ with and relative isomorphisms $\varphi_{i}:\left.X_{i} \rightarrow X\right|_{U_{i}}$ over $U_{i}$ such that $\varphi_{i} \circ \varphi_{i j}=\varphi_{j}$ over $U_{i j} ;$ if $F_{i}=F$, then $X$ is a fibre bundle with fibre $F$. These data are unique up to unique isomorphism, in fact, for any super-ringed space $Y / S$ and any $\psi_{i}:\left.X_{i} \rightarrow Y\right|_{U_{i}}$ over $U_{i}$, such that $\psi_{i} \circ \varphi_{i j}=\psi_{j}$ over $U_{i j}$, there is a unique morphism $\psi: X / S \rightarrow Y / S$ such that $\psi \circ \varphi_{i}=\psi_{i}$ over $U_{i}$.

Proof. The assertion will follow from Proposition 4.4 as soon as it is clear that the spaces $X_{i 0}$ glue to a paracompact Hausdorff space. Therefore, we drop all the subscripts ${ }_{0}$ and work in the category of topological spaces.

Let $X$ be the space obtained by gluing the $X_{i}$, and $p: X \rightarrow S$ the projection. Since $S$ is paracompact, we may assume that $\left(U_{i}\right)$ is locally finite, so $X$ has a locally finite open cover by paracompact Hausdorff spaces.

Hence, $X$ is paracompact, and to see that it is Hausdorff, it is sufficient to separate $x, y \in X$ with $p(x) \neq p(y)$ by disjoint open neighbourhoods. But this is trivial, since $S$ is Hausdorff.

Let us return to the inner homs. We first treat the linear case.
Lemma 4.49. Let $A$ be a Weil superalgebra $A$ and $V=\mathbb{R}^{p \mid q}$ as a linear supermanifold. Let $A \otimes V$ be the linear supermanifold whose underlying supervector space is the tensor product $A \otimes V$. Then $A \otimes V=\underline{\operatorname{Hom}}(\operatorname{Spec} A, V)$.
Proof. Applying Theorem 4.22, we compute, for any supermanifold $S$,

$$
\begin{aligned}
\operatorname{Hom}(S, A \otimes V) & =\operatorname{Hom}\left((A \otimes V)^{*}, \Gamma\left(\mathcal{O}_{S}\right)\right)=\left(\Gamma\left(\mathcal{O}_{S}\right) \otimes A \otimes V\right)_{0} \\
& =\operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{S}\right) \otimes A\right)=\operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{S \times \operatorname{Spec} A}\right)\right) \\
& =\operatorname{Hom}(S \times \operatorname{Spec} A, V) .
\end{aligned}
$$

In the last step, observe that Theorem 4.22 applies, in view of Lemma 4.44 and Remark 4.43.

Next, we pass to superdomains.
Lemma 4.50. Let $A$ be a Weil superalgebra, $V=\mathbb{R}^{p \mid q}$ as a linear supermanifold, and $U$ an open subspace of $V$. Then $U \times J_{A} \otimes V=\underline{\operatorname{Hom}}(\operatorname{Spec} A, U)$.

Proof. This follows by applying Lemma 4.25.
Construction 4.51. Let $A$ be a Weil superalgebra and $\varphi: X \rightarrow Y$ a morphism of supermanifolds such that $\underline{\operatorname{Hom}}(\operatorname{Spec} A, X)$ and $\underline{\operatorname{Hom}}(\operatorname{Spec} A, Y)$ exist. We define $\varphi_{A}: \underline{\operatorname{Hom}}(\operatorname{Spec} A, X) \rightarrow \underline{\operatorname{Hom}}(\operatorname{Spec} A, Y)$ to be the morphism given on $S$-points by:

$$
\varphi_{A}(s)=\varphi(s) \in_{S \times \operatorname{Spec} A} Y
$$

 is obviously functorial, i.e. if $\psi: Y \rightarrow Z$ satisfies the assumptions of $\varphi$, then $(\psi \circ \varphi)_{A}=\psi_{A} \circ \varphi_{A}$.

Recall the canonical morphisms $S \xrightarrow{j_{A}} S \times \operatorname{Spec} A \xrightarrow{p_{1}} S$. Composition with these defines canonical maps
$\underline{\operatorname{Hom}}(\operatorname{Spec} A, X)(S) \rightarrow X(S) \quad$ and $\quad X(S) \rightarrow \underline{\operatorname{Hom}}(\operatorname{Spec} A, X)$
which are natural in $S$ and therefore define morphisms

$$
p_{A}: \underline{\operatorname{Hom}}(\operatorname{Spec} A, X) \rightarrow X \quad \text { and } \quad s_{A}: X \rightarrow \underline{\operatorname{Hom}}(\operatorname{Spec} A, X)
$$

which satisfy $p_{A} \circ s_{A}=\mathrm{id}$.
The following diagram commutes:


Indeed, we compute for $s \in_{S} \underline{\operatorname{Hom}}(\operatorname{Spec} A, X)=X(S \times \operatorname{Spec} A)$,

$$
p_{A}\left(\varphi_{A}(s)\right)=\varphi(s) \circ j_{A}=\varphi\left(s \circ j_{A}\right)=\varphi\left(p_{A}(s)\right),
$$

and similarly for $s_{A}$. In particular, $\varphi_{A}$ is relative morphism over $\varphi$.
Proposition 4.52. For any Weil superalgebra $A$ and any supermanifold $X$, the inner hom $\underline{\operatorname{Hom}}(\operatorname{Spec} A, X)$ exists in $\operatorname{SMan}_{\mathbb{R}, \alpha}^{w}$. It has the structure of a fibre bundle over $X$, with typical fibre $J_{A} \otimes \mathbb{R}^{p \mid q}$, where $p \mid q=\operatorname{dim} X$. Thus, if $\operatorname{dim} A=r \mid s$, then $\operatorname{dim} \underline{\operatorname{Hom}(S p e c} A, X)=p r+q s \mid p s+q r$.
Proof of Proposition 4.52. Let $\left(U_{i}\right)$ be an open cover of $X$ by open subspaces isomorphic to superdomains. Then $\operatorname{Hom}\left(\operatorname{Spec} A, U_{i}\right)$ exists and equals $U_{i} \times J_{A} \otimes \mathbb{R}^{p \mid q}$ by Lemma 4.50. Applying Construction 4.51, we obtain isomorphisms $\left.\left.\underline{\operatorname{Hom}}\left(\operatorname{Spec} A, U_{j}\right)\right|_{U_{i j}} \rightarrow \underline{\operatorname{Hom}}\left(\operatorname{Spec} A, U_{i}\right)\right|_{U_{i j}}$ over $U_{i j}$. Then Proposition 4.48 implies our claim.
Definition 4.53. Let $A$ be a Weil superalgebra. For any supermanifold, the fibre bundle $p_{A}: \underline{\operatorname{Hom}}(\operatorname{Spec} A, X) \rightarrow X$ is called the $A$-Weil bundle of $X$ and denoted $T_{A} X$.

The functor $T_{A}$ which maps $X \mapsto T_{A} X$ and morphisms $\varphi: X \rightarrow Y$ to morphisms $T_{A} \varphi=\varphi_{A}: T_{A} X \rightarrow T_{A} Y$ over $\varphi$ is called the $A$-Weil functor.
4.54. Let $V$ be a linear supermanifold. For the underlying super-vector space, the structure maps of addition $V \times V \rightarrow V$ and multiplication by scalars $\mathbb{R} \times V \rightarrow V$ give even linear maps $V \otimes V \rightarrow V$ and $\mathbb{R} \otimes V \rightarrow V$. The transpose of these maps defines elements of $\operatorname{Hom}\left(V^{*},(V \otimes V)^{*}\right)$ and $\operatorname{Hom}\left(V^{*},(\mathbb{R} \otimes V)^{*}\right)$, respectively. Since $(W \otimes V)^{*}=W^{*} \otimes V^{*} \subset \Gamma\left(\mathcal{O}_{W \times V}\right)$ for $W=\mathbb{R}$ or $W=V$, we obtain morphisms $V \times V \rightarrow V$ and $\mathbb{R} \times V \rightarrow V$.

In particular, for any super-ringed space $S, V(S)$ has the structure of a $\mathbb{R}(S)$-module. Let $U$ be an open subspace of $V$. For $v \in_{S} V$, we say that $v \in_{S} U$ if $v=j_{U}\left(v^{\prime}\right)$ for some $v^{\prime} \in_{S} U$. Since (being a monomorphism) $j_{U}$ is injective on $S$-points, $v^{\prime}$ is then unique, and this makes sense.
Lemma 4.55. Let $U$ be an open subspace of $V$. There exists a largest open neighbourhood $U^{[1]}$ of $U \times 0 \times V$ in $U \times \mathbb{R} \times V$ such that for any super-ringed space $S$,

$$
u+\varepsilon v \in_{S} U \text { for all }(u, \varepsilon, v) \in_{S} U^{[1]}
$$

Proof. For any morphism $\varphi: X \rightarrow Y$ and any open subspace $V \subset Y$, let $\varphi^{-1}(V) \subset X$ denote the open subspace correpsonding to the open set $\varphi_{0}^{-1}\left(V_{0}\right)$. Then $\varphi$ induces a morphism $\varphi: \varphi^{-1}(V) \rightarrow V(c f$. Lemma 4.25).

In our present situation, consider the morphism $\varphi$ defined on points by $V(S) \times \mathbb{R}(S) \times V(S) \rightarrow V(S):(u, \varepsilon, v) \mapsto u+\varepsilon v$. The $U^{[1]}=\varphi^{-1}(U)$. Since $\varphi$ maps $(u, 0, v)$ to $u$, this is an open neighbourhood $U \times 0 \times V$, and it obviously contains any other open neighbourhood satisfying the assumption of the lemma.

The following may be viewed as a generalised form of Hadamard's lemma.
Lemma 4.56. Let $X$ be a supermanifold, $V$ a linear supermanifold, and $U$ an open neighbourhood of 0 in $\mathbb{R}$. For any morphism $f: X \times U \rightarrow V$ such that $f(x, 0)=0$ for all $x \in_{S} X$, there is a unique morphism $g: X \times U \rightarrow V$ such that

$$
f(x, \varepsilon)=\varepsilon g(x, \varepsilon) \quad \text { for all }(x, \varepsilon) \in_{S} X \times U
$$

Proof. First, we prove the uniqueness. It is convenient to use the topology on $X(S)$ consisting of all $W(S)$ where $W$ is an open subspace of $X$. For any morphism $\varphi: X \rightarrow Y, \varphi_{S}: X(S) \rightarrow Y(S)$ is continuous. Indeed, $\varphi_{S}^{-1}(V(S))=\varphi^{-1}(V)(S)$. We remark that the topology on $X(S) \times U(S)$ thus introduced coincides with the product topology of $X(S)$ and $U(S)$.

Let $g, g^{\prime}$ satisfy the assumption in the assertion. If $\varepsilon \in_{S}(U \backslash 0)$, then $\varepsilon$ is an invertible element of $\mathbb{R}(S)$. Thus, $g(x, \varepsilon)=g^{\prime}(x, \varepsilon)$ for all $x \in_{S} X$ and $\varepsilon \in_{S}(U \backslash 0)$. But $(U \backslash 0)(S)$ is dense in $U(S)$, so $X(S) \times(U \backslash 0)(S)$ is dense in $X(S) \times U(S)$, and by continuity, $g$ and $g^{\prime}$ coincide on $X(S) \times U(S)$. Since $S$ was arbitrary, $g=g^{\prime}$.

For the existence, in view of Theorem 4.22, by composing with linear forms on $V$, we may assume $V=\mathbb{R}^{1 \mid 1}$. So, in fact, we are considering a superfunction $f$. Moreover, by the uniqueness, we may assume that $X \subset \mathbb{R}^{p \mid q}$ is a superdomain. Thus, $X \times U$ is an open subspace of $\mathbb{R}^{p+1 \mid q}$, and $f=\sum_{I} f_{I} \xi^{I}$ for some $f_{I} \in \mathcal{C}^{\infty}\left(X_{0} \times U_{0}\right)$ where $f_{I}(x, 0)=0$ for all $x \in X_{0}$. Thus, one may set $g=\sum_{I} g_{I} \xi^{I}$ where $g_{I}(x, \varepsilon)=\left.\int_{0}^{1} \partial_{s} f_{I}(x, s)\right|_{s=\varepsilon t} d t$.
Definition 4.57. Let $U$ be an open subspace of a linear supermanifold $V$, and $W$ be another linear supermanifold. Let $f: U \rightarrow W$ be a morphism and $U^{[1]}$ a neighbourhood as in Lemma 4.55.

By Lemma 4.56, there exists a unique morphism $f^{[1]}: U^{[1]} \rightarrow W$ such that

$$
f(u+\varepsilon v)-f(u)=\varepsilon f^{[1]}(u, \varepsilon, v) \quad \text { for all }(u, \varepsilon, v) \in_{S} U^{[1]}
$$

and all super-ringed spaces $S$. (This follows by covering $U^{[1]}$ by open subpaces of the form $U_{\alpha} \times V_{\alpha} \times W_{\alpha}$ where $\left(U_{\alpha}\right)$ is an open cover of $U$, and $V_{\alpha}$, $W_{\alpha}$ are open neighbourhoods of 0 in $\mathbb{R}$ and $V$, respectively.)

We let $d f: U \times V \rightarrow W$ be the morphism defined by

$$
d f(u) v=d f(u)(v)=f^{[1]}(u, 0, v) \quad \text { for all } u \in_{S} U, v \in_{S} V
$$

Inductively, define $d^{n} f: U \times V^{n} \rightarrow W$ by

$$
d^{n+1} f(u)\left(v_{1}, \ldots, v_{n+1}\right)=d\left(d^{n} f(\cdot)\left(v_{1}, \ldots, v_{n}\right)\right)\left(u, v_{n+1}\right)
$$

for all $u \in_{S} U, v_{1}, \ldots, v_{n+1} \in_{S} V$.
The morphism $d^{n} f$ is called the $n^{\text {th }}$ derivative of $f$.

Proposition 4.58. Let $V, W$ be linear supermanifolds, $U \subset V$ an open subspace, and $f: U \rightarrow W$ a morphism. The $n^{t h}$ derivative $d^{n} f$ is symmetric n-linear, that is,

$$
\begin{gathered}
d^{n} f(u)\left(\ldots, \lambda v+v_{j}, \ldots\right)=\lambda d^{n} f(u)(\ldots, v, \ldots)+d^{n} f(u)\left(\ldots, v_{j}, \ldots\right) \\
d^{n} f(u)\left(\ldots, v_{i}, \ldots, v_{j}, \ldots\right)=d^{n} f(u)\left(\ldots, v_{j}, \ldots, v_{i}, \ldots\right)
\end{gathered}
$$

for all $u \in_{S} U, v, v_{i}, v_{j} \in_{S} V, \lambda \in_{S} \mathbb{R}$, and any super-ringed space $S$.
Proof. It is sufficient to prove the linearity for the first derivative $d f$. To that end, note

$$
\begin{aligned}
f(u+\varepsilon(\lambda v+w)) & =f(u+\varepsilon(\lambda v+w))-f(u+\varepsilon w)+f(u+\varepsilon w) \\
& =\varepsilon \lambda f^{[1]}(u+\varepsilon w, \varepsilon \lambda, v)+\varepsilon f^{[1]}(u, \varepsilon, w)+f(u)
\end{aligned}
$$

so

$$
\lambda f^{[1]}(u+\varepsilon w, \varepsilon \lambda, v)+f^{[1]}(u, \varepsilon, w)=f^{[1]}(u, \varepsilon, \lambda v+w)
$$

from which the assertion follows by setting $\varepsilon=0$.
We prove the symmetry in the case of the second derivative. The general case then follows by a trivial induction. There is a unique $g$ such that

$$
\varepsilon g\left(u, \delta, v_{1}, \varepsilon, v_{2}\right)=f^{[1]}\left(u+\varepsilon v_{2}, \delta, v_{1}\right)-f^{[1]}\left(u, \delta, v_{1}\right)
$$

in particular, $d^{2} f(u)\left(v_{1}, v_{2}\right)=g\left(u, 0, v_{1}, 0, v_{2}\right)$.
We compute

$$
\begin{aligned}
& \delta\left(f^{[1]}\left(u+\varepsilon v_{2}, \delta, v_{1}\right)-f^{[1]}\left(u, \delta, v_{1}\right)\right) \\
& \left.\quad=f\left(u+\delta v_{1}+\varepsilon v_{2}\right)\right)-f\left(u+\varepsilon v_{2}\right)-f\left(u+\delta v_{1}\right)+f(u)
\end{aligned}
$$

so that

$$
\begin{aligned}
\varepsilon \delta g\left(u, \delta, v_{1}, \varepsilon, v_{2}\right) & =\varepsilon \delta\left(f^{[1]}\left(u+\varepsilon v_{2}, \delta, v_{1}\right)-f^{[1]}\left(u, \delta, v_{1}\right)\right) \\
& =\varepsilon \delta\left(f^{[1]}\left(u+\delta v_{1}, \varepsilon, v_{2}\right)-f^{[1]}\left(u, \varepsilon, v_{2}\right)\right) \\
& =\varepsilon \delta g\left(u, \varepsilon, v_{2}, \delta, v_{1}\right)
\end{aligned}
$$

and in particular, $d^{2} f(u)\left(v_{1}, v_{2}\right)=d^{2} f(u)\left(v_{2}, v_{1}\right)$.
Proposition 4.59. Let $V_{1}, V_{2}$, $W$ be linear supermanifolds, $U_{j} \subset V_{j}$ open subspaces, and $f: U_{1} \rightarrow U_{2}, g: U_{2} \rightarrow W$ morphisms. Then

$$
d(g \circ f)(u) v=d g(f(u)) d f(u) v \quad \text { for all } \quad u \in_{S} U_{1}, v \in_{S} V_{1}
$$

Proof. We compute
$g(f(u+\varepsilon v))=g\left(f(u)+\varepsilon f^{[1]}(u, \varepsilon, v)\right)=\varepsilon g^{[1]}\left(f(u), \varepsilon, f^{[1]}(u, \varepsilon, v)\right)+g(f(u))$,
so $(g \circ f)^{[1]}(u, \varepsilon, v)=g^{[1]}\left(f(u), \varepsilon, f^{[1]}(u, \varepsilon, v)\right)$, which proves the claim.
Definition 4.60. Let $V, W$ be linear supermanifolds, $U \subset V$ an open subspace and $f: U \rightarrow W$ a morphism. Define, for $n \in \mathbb{N}$, the $n^{t h}$ Taylor polynomial $P^{n} f: U \times V \rightarrow W$ of $f$ by

$$
P^{n} f(u, v)=\sum_{j=0}^{n} \frac{1}{j!} d^{n} f(u)(v, \ldots, v) \quad \text { for all } u \in_{S} U, v \in_{S} V
$$

Proposition 4.61. Let $V, W$ be linear supermanifolds, $U \subset V$ an open subspace and $f: U \rightarrow W$ a morphism. There exists a unique morphism $f^{[n+1]}: U^{[1]} \rightarrow W$ such that

$$
f(u+\varepsilon v)-P^{n} f(u, \varepsilon v)=\varepsilon^{n+1} f^{[n+1]}(u, \varepsilon, v) \quad \text { for all } \quad(u, \varepsilon, v) \in_{S} U^{[1]}
$$

Proof. First, we prove, by induction on $n$, that there exist morphisms some $b_{j}: U \times V \rightarrow W, f^{[n+1]}: U^{[1]} \rightarrow W$, such that

$$
\begin{equation*}
f(u+\varepsilon v)=\sum_{j=0}^{n} \varepsilon^{j} b_{j}(u, v)+\varepsilon^{n+1} f^{[n+1]}(u, \varepsilon, v) \quad \text { for all } \quad(u, \varepsilon, v) \in_{S} U^{[1]} \tag{4.2}
\end{equation*}
$$

The claim is clear for $n=0$. So, let $n \geqslant 1$ and assume that the claim has been proved for $n-1$. Define $b_{n}$ by $b_{n}(u, v)=f^{[n]}(u, 0, v)$. Then

$$
f(u+\varepsilon v)-\sum_{j=0}^{n} \varepsilon^{j} b_{j}(u, v)=\varepsilon^{n}\left(f^{[n]}(u, \varepsilon, v)-f^{[n]}(u, 0, v)\right)
$$

This proves our first assertion, in view of Lemma 4.56.
To complete the proof, consider the operator $L_{p}$, defined for any morphism $g: V \rightarrow W$ inductively by

$$
L_{1} g(x, v)=g(x+v)-g(x), L_{p} g(x, v)=L_{p-1} g(x+v, v)-L_{p-1} g(x, v)
$$

Then

$$
L_{p} g(x, v)=\sum_{j=0}^{p}(-1)^{j}\binom{n}{j} g(x+(p-j) v)
$$

as is easily proved by induction.
The operators $L_{p}$ have following property.
Lemma 4.62. We have $L_{n} f(u+\varepsilon \cdot)(0, v)=L_{n} f(u, \varepsilon v)=\varepsilon^{n} g(u, \varepsilon, v)$ for some unique morphism $g$ such that $g(u, 0, v)=d^{n} f(u)(v, \ldots, v)$.
Proof. First, we notice that $L_{n} f(u+\varepsilon \cdot)(0, v)=L_{n} f(u, \varepsilon v)$ by definition. Next, we introduce operators $\tilde{L}_{n}$ inductively by $\tilde{L}_{1}=L_{1}$ and

$$
\tilde{L}_{n+1} h\left(u, v_{1}, \ldots, v_{n}\right)=\tilde{L}_{n} h\left(u+v_{n}, v_{1}, \ldots, v_{n-1}\right)-\tilde{L}_{n} h\left(u, v_{1}, \ldots, v_{n}\right)
$$

In particular, $L_{n} h(u, v)=\tilde{L}_{n} h(u, v, \ldots, v)$.
We prove by induction:

$$
\tilde{L}_{n} f\left(u, \varepsilon_{1} v_{1}, \ldots, \varepsilon_{n} v_{n}\right)=\varepsilon_{1} \cdots \varepsilon_{n} g_{n}\left(u, \varepsilon_{1}, v_{1}, \ldots, \varepsilon_{n}, v_{n}\right)
$$

for some morphism $g_{n}$ such that

$$
g_{n}\left(u, 0, v_{1}, \ldots, 0, v_{n}\right)=d f^{n}(u)\left(v_{1}, \ldots, v_{n}\right)
$$

From this, the assertion will follow, since uniqueness is obvious.
The statement is obvious for $n=1$. Let is be proven for $n-1$. Then

$$
\begin{aligned}
\tilde{L}_{n} f\left(u, \varepsilon_{1} v_{1}, \ldots\right) & =\tilde{L}_{n-1}\left(u+\varepsilon_{n} v_{n}, \varepsilon_{1}, v_{1}, \ldots\right)-\tilde{L}_{n-1}\left(u, \varepsilon_{1}, v_{1}, \ldots\right) \\
& =\varepsilon_{1} \cdots \varepsilon_{n-1}\left(g_{n-1}\left(u+\varepsilon_{n} v_{n}, \ldots\right)-g_{n-1}(u, \ldots)\right) \\
& =\varepsilon_{1} \cdots \varepsilon_{n} \cdot g_{n}\left(u, \varepsilon_{1}, v_{1}, \ldots, \varepsilon_{n}, v_{n}\right)
\end{aligned}
$$

for some $g_{n}$ such that

$$
g_{n}\left(u, \varepsilon_{1}, v_{1}, \ldots, \varepsilon_{n-1}, 0, v_{n}\right)=d\left[g_{n-1}\left(\cdot, \varepsilon_{1}, v_{1}, \ldots, \varepsilon_{n-1}, v_{n-1}\right)\right](u)\left(v_{n}\right)
$$

From this, the statement follows.
We will also need the following combinatorial identity.
Lemma 4.63. For $n \geqslant 1$ and $m \geqslant n \geqslant k \geqslant 0$, we have

$$
\sum_{j=0}^{k}\binom{n}{j}(-1)^{j}(m-j)^{k}=\delta_{k n} n!
$$

Proof. We prove the claim by induction on $n$. For $n=1, k=0$, we have $m^{0}-(m-1)^{0}=0$, and for $k=1$, we have

$$
m^{1}-(m-1)^{1}=1
$$

Assuming the claim for $n \geqslant 1, m \geqslant n \geqslant k$, let $m \geqslant n+1 \geqslant k \geqslant 0$. Then

$$
\begin{aligned}
& \sum_{j=0}^{n+1}\binom{n+1}{j}(-1)^{j}(m-j)^{k} \\
& \quad=\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(m-j)^{k}-\sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(m-1-j)^{k}=(*)
\end{aligned}
$$

Now,

$$
(m-j)^{k}-(m-1-j)^{k}=\sum_{i=0}^{k-1}\binom{k}{i}(m-1-j)^{i}
$$

so

$$
\begin{aligned}
(*) & =\sum_{i=0}^{k-1}\binom{k}{i} \sum_{j=0}^{n}\binom{n}{j}(-1)^{j}(m-1-j)^{i} \\
& =\sum_{i=0}^{k-1}\binom{k}{i} \delta_{i n} n!=\binom{k}{k-1} \delta_{k-1, n} n!=\delta_{k, n+1}(n+1)!
\end{aligned}
$$

This proves the claim.
Proof of Proposition 4.61 (continued). Fix $n$ and any representation of the form (4.2). We now prove by induction on $\ell \leqslant n$ that

$$
b_{j}(u, v)=\frac{1}{j!} d^{j} f(u)(v, \ldots, v) \quad \text { for all } j \leqslant \ell
$$

The case of $\ell=0$ is easy deduced by setting $\varepsilon=0$. So let $\ell \geqslant 0$, and assume the assertion has been proved for $\ell-1$. If $g$ is homogeneous of degree $k \leqslant \ell$, i.e. $g(q x)=q^{k} g(x)$ for all $q \in \mathbb{N}$, then

$$
L_{\ell} g(m v, v)=\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^{j}(n+m-j)^{k} g(v)=\delta_{k \ell} \ell!\cdot g(v)
$$

by Lemma 4.63 . Hence, we have

$$
L_{\ell} f(u+\varepsilon \cdot)(0, v)=\varepsilon^{\ell} L_{\ell} b_{\ell}(u, \cdot)(0, v)+\varepsilon^{\ell+1} g(u, \varepsilon, v)
$$

where

$$
g(u, \varepsilon, v)=\sum_{j=\ell+1}^{n} \varepsilon^{j-\ell-1} L_{\ell} b_{j}(u, \cdot)(0, v)+\varepsilon^{n+1} L_{n+1} f^{[n+1]}(u, \varepsilon, \cdot)(0, v)
$$

From Lemma 4.62, we conclude that

$$
L_{\ell} b_{\ell}(u, \cdot)(0, v)=d^{\ell} f(u)(v, \ldots, v) .
$$

In particular,
$b_{\ell}(u, \cdot)$ is homogeneous of degree $\ell$, so that $L_{\ell} b_{\ell}(u, v)=\ell!\cdot b_{\ell}(u, v)$ by the above. This proves the proposition.
Proposition 4.64. Let $V_{1}, \ldots, V_{n}$ and $W$ be linear supermanifolds. Any even n-linear map $A: V_{1} \times V_{n} \rightarrow W$ of the underlying super vector spaces gives rise to a unique morphism $\varphi_{A}: V_{1} \times V_{n} \rightarrow W$ such that $\varphi_{A}^{*}(\mu)=$ $\mu \circ A \in V_{1}^{*} \otimes \cdots \otimes V_{n}^{*} \subset \Gamma\left(\mathcal{O}_{V_{1} \times \cdots \times V_{n}}\right)$.

On S-points, $\varphi_{A}$ is given as follows: Given $\left(v_{1}, \ldots, v_{n}\right) \in_{S} \prod_{j=1}^{n} V_{j}$, we have $v_{j} \in\left(\Gamma\left(\mathcal{O}_{S}\right) \otimes V_{j}\right)_{0}$, so $v_{j}=\sum_{i} s_{j i} \otimes w_{j i}$ for some $s_{j i} \in \Gamma\left(\mathcal{O}_{S}\right), w_{j i} \in V_{j}$. Then

$$
\begin{equation*}
\varphi_{A}\left(v_{1}, \ldots, v_{n}\right)=\sum_{i_{1}, \ldots, i_{n}}(-1)^{\sum_{k>\ell}\left|w_{k i_{k}}\right|\left|w_{\varepsilon_{i}}\right|} s_{1 i_{1}} \cdots s_{n i_{n}} A\left(w_{1 i_{1}}, \ldots, w_{n i_{n}}\right) \tag{4.3}
\end{equation*}
$$

Proof. The first statement is obvious, so we need to prove (4.3). We treat the linear case, the general case then being an exercise in sign bookkeeping.

Thus, $n=1$, and $V=V_{1}$. By the first part, we may assume $W=\mathbb{R}^{111}$, so that we are dealing with superfunctions, and $A \in V^{*}, \varphi_{A}=A$. Let $\varphi \in \Gamma\left(\mathcal{O}_{V}\right)$ be defined on points $s=\sum_{i} s_{i} \otimes w_{i} \in_{S} V$ by $\varphi(s)=\sum_{i} s_{i} A\left(w_{i}\right)$. But this is just $s^{*}(A)=\varphi_{A}(s)$, as one sees by running through the isomorphism $\operatorname{Hom}\left(V^{*}, \Gamma\left(\mathcal{O}_{S}\right)\right)=\left(\Gamma\left(\mathcal{O}_{S}\right) \otimes V\right)_{0}$. Hence, $\varphi=\varphi_{A}$.
Proposition 4.65. Let $X$ be a supermanifold and $V_{1}, \ldots, V_{n}, W$ be linear supermanifolds. There is a natural bijection between the set of morphisms $\varphi: X \times V_{1} \times \cdots \times V_{n} \rightarrow W$ such that

$$
\begin{equation*}
\varphi\left(x, v_{1}, \ldots, \lambda v_{j}+w, \ldots\right)=\lambda \varphi\left(x, v_{1}, \ldots, v_{j}, \ldots\right)+\varphi\left(x, v_{1}, \ldots, w, \ldots\right) \tag{4.4}
\end{equation*}
$$

for all $x \in_{S} X, \lambda \in_{S} \mathbb{R}_{\alpha},\left(v_{1}, \ldots, v_{n}\right) \in_{S} V_{1} \times \cdots \times V_{n}, w_{j} \in_{S} V_{j}$, all $j=1, \ldots, n$, and the set

$$
\operatorname{Hom}\left(W^{*}, \Gamma\left(\mathcal{O}_{X}\right) \otimes \bigotimes_{j=1}^{n} V_{j}^{*}\right)_{\alpha}=\left(\Gamma\left(\mathcal{O}_{X}\right) \otimes \underline{\operatorname{Hom}}\left(V_{1} \otimes \cdots \otimes V_{n}, W\right)\right)_{0, \alpha}
$$

This set is also equal to $\operatorname{Hom}\left(X, \underline{\operatorname{Hom}}\left(V_{1} \otimes \cdots \otimes V_{n}, W\right)\right)$ where the supervector space $\operatorname{Hom}\left(V_{1} \otimes \cdots \otimes V_{n}, W\right)$ is considered as a linear supermanifold.

Proof. In view of Theorem 4.22, $\operatorname{Hom}\left(W^{*}, \Gamma\left(\mathcal{O}_{X \times V_{1} \times \cdots V_{n}}\right)\right)_{\alpha}$ is the same as the morphisms $X \times V_{1} \times \cdots V_{n} \rightarrow W$, so what we need to prove is that Equation (4.4) singles out the linear maps with values in $\Gamma\left(\mathcal{O}_{X}\right) \otimes V_{1}^{*} \otimes \cdots \otimes V_{n}^{*}$. That the elements of the latter set satisfy the equation is straightforward.

Conversely, again by Theorem 4.22, we may assume $W=\mathbb{R}^{1 \mid 1}$, so that the statement is about superfunctions. Since the $V_{j}$ are finite-dimensional, it is easy to see that the statement is local. Thus, we assume that $X$ is a superdomain in with coordinates $(x, \xi)$. We consider the case of $n=1$, the general case being then an exercise in notation.

Thus $V=V_{1}$, and we consider on this space coordinates $(y, \eta)$. Let $\varphi \in \Gamma\left(\mathcal{O}_{X \times V}\right)$ satisfy Equation (4.4). We may write $\varphi=\sum_{I J} f_{I J} \xi^{I} \eta^{J}$
where $f_{I J} \in \mathcal{C}^{\varpi}\left(X_{0} \times V_{0, \alpha}\right)$. We let $f_{J}=\sum_{I} f_{I J} \xi^{I}$ and consider this as a $\Gamma\left(\mathcal{O}_{X}\right)$-valued function on $V_{0, \alpha}$. The assumption on $\varphi$ implies

$$
f_{J}(v)=|\lambda|^{|J|-1} f_{J}(\lambda v) \text { for all } \lambda \neq 0, v \in V_{0}
$$

Thus, $f_{J}=0$ unless $|J| \neq 0,1$. Moreover, $f_{\varnothing}$ is linear, so $f_{J} \in \Gamma\left(\mathcal{O}_{X}\right) \otimes V^{*}$, and $f_{J}$ is constant for $|J|=1$, so $f_{J} \in \Gamma\left(\mathcal{O}_{X}\right)$ in this case. But then we have $\varphi=f_{\varnothing}+\sum_{i} f_{i} \eta_{i} \in \Gamma\left(\mathcal{O}_{X}\right) \otimes V^{*}$, as claimed.
Corollary 4.66. In the situation of Proposition 4.65, assume given a morphism $\varphi: X \times \prod_{j=1}^{n} V_{j} \rightarrow W$ satisfying Equation (4.4).

If $A$ is any finite-dimensional $\mathbb{R}_{\alpha}$-superalgebra, then $\varphi$ has a canonical $A$-multilinear extension $\operatorname{id}_{A} \otimes \varphi: X \times \prod_{j=1}^{n}\left(A \otimes V_{j}\right) \rightarrow A \otimes W$.
Theorem 4.67. Let $U_{j} \subset V_{j}, j=1,2$, be open subspaces of linear supermanifolds, $A$ a Weil superalgebra, and $\varphi: U_{1} \rightarrow U_{2}$ a morphism. Then

$$
T_{A} \varphi: T_{A} U_{1}=U_{1} \times J_{A} \otimes V_{1} \rightarrow T_{A} U_{2}=U_{2} \times J_{A} \otimes V_{2}
$$

is given on $S$-points by

$$
T_{A} \varphi(u, v)=\sum_{k=0}^{\infty} \frac{1}{k!} d^{k} \varphi(u)(v, \ldots, v) \quad \text { for all } \quad(u, v) \in_{S} U_{1} \times J_{A} \otimes V_{1}
$$

where the summands on the right hand side are defined by $A$-linear extension, v. Corollary 4.66.

Example 4.68. Let $\mathbb{D}_{k}=\mathbb{R}[\varepsilon] /\left(\varepsilon^{k+1}\right)$ where the indeterminate $\varepsilon$ is even. Then in the setup of Theorem 4.67, $T_{A} U_{1}=U_{1} \times \prod_{j=1}^{k} \varepsilon^{j} V_{1}$, and

$$
T_{A} \varphi(u, v)=\varphi(u)+\varepsilon \cdot d \varphi(u)(v)+\cdots+\frac{\varepsilon^{k}}{k!} \cdot d \varphi(u)(v, \ldots, v)
$$

is the formal Taylor polynomial of $\varphi$, of order $k$. One is tempted to identify this with $P^{k} \varphi, v$. Definition 4.60.

The example motivates the following definition.
Definition 4.69. Let $\mathbb{D}=\mathbb{D}_{1}$, the ring of dual numbers. For any supermanifold $X$, let $T X=T_{\mathbb{D}} X$, and call this the tangent bundle of $X$. Furthermore, let $J^{k} X=T_{\mathbb{D}_{k}} X$, and call this the $k$-th jet bundle of $X$. For $\mathbb{D}^{k}=\mathbb{D} \otimes \cdots \otimes \mathbb{D}$, let $T^{k} X=T_{\mathbb{D}^{k}} X$. This is called the $k$-th tangent bundle of $X$.

If $\varphi: X \rightarrow Y$ is a morphism of supermanifolds, $T_{A} \varphi$ is called, for $A=\mathbb{D}$, $\mathbb{D}_{k}$, and $\mathbb{D}^{k}$, respectively, the tangent morphism, the $k$-th jet morphism, and the $k$-th tangent morphism of $\varphi$.
Example 4.70. Another basic example of a Weil superalgebra is given by $A=\Lambda\left(\mathbb{R}^{0 \mid s}\right)^{*}$. In this case, Spec $A=\mathbb{R}^{0 \mid s}$ is a supermanifold. One has

$$
T_{A} U=U \times \mathbb{R}^{q s \mid p s}
$$

In particular, for $s=1$, we have $T_{A} U=U \times \mathbb{R}^{q \mid p}$, so the bundle $T_{A} X$ is derived from $T X$ just by changing the parity of the linear fibre. It is called the odd tangent bundle in the literature, and denoted by ПTX.

For an ordinary manifold $X($ viz. $q=0)$, the supermanifold $\Pi T X$ has the structure sheaf $\Omega_{X}^{\bullet} \otimes \mathbb{R}$ of $\mathbb{R}$-valued differential forms on $X$. For this reason, the superfunctions on $\Pi T X$ are a generalisation of differential forms
to supermanifolds. They are called pseudodifferential forms. We will discuss them at length below.

Proposition 4.71. Let $A$ be a Weil superalgebra. The Weil functor $T_{A}$ enjoys the following properties:
(1). $T_{A}$ commutes with finite products: in particular, $T_{A} G$ is a Lie supergroup for any Lie supergroup $G$;
(2). if $U$ is an open subspace of $X$, then $T_{A} U$ is an open subspace of $T_{A} X$;
(3). if $B$ is another Weil superalgebra, then $T_{A \otimes B} X=T_{A}\left(T_{B} X\right)$, where the isomorphism is natural; in particular, $T_{A} \circ T_{B} \cong T_{B} \circ T_{A}$.

Proof. (1). The statement about supergroups follows from the statement about products. By definition, in the case of binary products, this may be checked locally, where it is obvious. For the empty product, we have

$$
\operatorname{Hom}\left(S, T_{A}(*)\right)=\operatorname{Hom}(S \times \operatorname{Spec} A, *)=*,
$$

so $T_{A}(*)=*$.
(2). This is clear by construction.
(3). We observe $\operatorname{Spec}(A \otimes B)=\operatorname{Spec} A \times \operatorname{Spec} B$, so

$$
\begin{aligned}
\operatorname{Hom}\left(S, T_{A \otimes B} X\right) & =\operatorname{Hom}(S \times \operatorname{Spec} A \times \operatorname{Spec} B, X) \\
& =\operatorname{Hom}\left(S \times \operatorname{Spec} A, T_{B} X\right)=\operatorname{Hom}\left(S, T_{A}\left(T_{B} X\right)\right)
\end{aligned}
$$

for any supermanifold $S$.
Corollary 4.72. Any natural transformation $T_{A} \rightarrow T_{B}$ (with $A$, $B$ Weil superalgebras) is uniquely determined by its value at $X=\mathbb{R}$.

Proof. Indeed, let $\sigma, \tau: T_{A} \rightarrow T_{B}$ be natural. We first show that they are determined by their values on $\mathbb{R}^{1 \mid 1}$. So, assume that $\sigma_{\mathbb{R}^{1 \mid 1}}=\tau_{\mathbb{R}^{1 \mid 1}}$.

Since $T_{A}$ and $T_{B}$ commute with finite products, the naturality of $\sigma$ and $\tau$ gives $\sigma_{X}=\tau_{X}$ for $X=\mathbb{R}^{1 \mid 0}$ and $X=\mathbb{R}^{0 \mid 1}$, and then for $X=\mathbb{R}^{p \mid q}$ where $p$ and $q$ are arbitrary. By item (2) of Proposition 4.71, this equality extends to the case of an open subspace $X$ of $\mathbb{R}^{p \mid q}$, and thus, to an arbitrary supermanifold, in view of Proposition 4.4.

This proves the claim under the assumption of $\sigma_{X}=\tau_{X}$ for $X=\mathbb{R}^{1 \mid 1}$. Naturality and item (3) of Proposition 4.71 show that $T_{C} \tau_{X}=\tau_{T_{C} X}$ for any Weil superalgebra $C$. However, for $C=\bigwedge \mathbb{R}^{0 \mid 1}$, we have $T_{C} \mathbb{R}^{1 \mid 0}=\mathbb{R}^{1 \mid 1}$, by Lemma 4.49. Then $\tau_{\mathbb{R}^{1 \mid 1}}=T_{C} \tau_{\mathbb{R}^{1 \mid 0}}$, and this finally proves the assertion.

A nice property of Weil functors is their behaviour under 'base change'.
Construction 4.73. Let $f: A \rightarrow B$ is an (even and unital) algebra morphism of Weil superalgebras. It induces a morphism of ringed spaces $\operatorname{Spec} B \rightarrow \operatorname{Spec} A$, given by $\left(\mathrm{id}_{*}, f\right)$. Denote this morphism by $f^{*}$.

Let $X$ be a supermanifold. We define $T_{f, X}: T_{A} X \rightarrow T_{B} X$ on $S$-points by

$$
T_{f, X}(s)=s \circ\left(\operatorname{id}_{S} \times f^{*}\right) \quad \text { for all } s \in\left(T_{A} X\right)(S)=X(S \times \operatorname{Spec} A)
$$

Proposition 4.74. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be morphisms of Weil superalgebras.
(1). The collection $\left(T_{f, X}\right)$, for all supermanifolds $X$, defines a natural transformation $T_{f}: T_{A} \rightarrow T_{B}$, and $T_{g \circ f}=T_{g} \circ T_{f}$.
(2). If $f$ is surjective, then $T_{f, X}: T_{A} X \rightarrow T_{B} X$ is a fibre bundle with fibre $T_{\text {ker } f} X$, for any supermanifold $X$ of pure dimension. If $f$ is injective, then $T_{f, X}$ is locally isomorphic $\left(\mathrm{id}_{T_{A} X}, s_{\text {coker } f} \circ p_{A}\right): T_{A} X \rightarrow T_{A} X \times T_{\text {coker } f} X$.
(3). The map $f \mapsto T_{f}$ is an isomorphism $\operatorname{Hom}(A, B) \rightarrow \operatorname{Hom}\left(T_{A}, T_{B}\right)$. Here, we consider $T_{A}, T_{B}$ as functors with values in supermanifolds, and do not a priori assume that natural transformations consist of bundle morphisms.

Proof. (1). Let $\varphi: X \rightarrow Y$ be a morphism of supermanifolds. Then

$$
\begin{aligned}
\left(T_{f, Y} \circ T_{A} \varphi\right)(s) & =T_{f, Y}(\varphi(s))=\varphi(s) \circ\left(\operatorname{id}_{S} \times f^{*}\right) \\
& =\varphi\left(s \circ\left(\operatorname{id}_{S} \times f^{*}\right)\right)=T_{B} \varphi\left(s \circ\left(\operatorname{id}_{S} \times f^{*}\right)\right)=\left(T_{B} \varphi \circ T_{f, X}\right)(s)
\end{aligned}
$$

for all $s \in T_{A} X(S)=X(S \times \operatorname{Spec} A)$. Hence, $T_{f}$ is a natural transformation.
Next, compute

$$
\begin{aligned}
\left(T_{g, X} \circ T_{f, X}\right)(s) & =T_{g, X}\left(s \circ\left(\operatorname{id}_{S} \times f^{*}\right)\right)=s \circ\left(\operatorname{id}_{S} \times f^{*}\right) \circ\left(\operatorname{id}_{S} \times g^{*}\right) \\
& =s \circ\left(\operatorname{id}_{S} \times(g \circ f)^{*}\right)=T_{g \circ f, X}(s)
\end{aligned}
$$

for all $s \in\left(T_{A} X\right)(S)=X(S \times \operatorname{Spec} A)$.
(2). Both statements are local. Let $X=U$ be an open subspace of $V=\mathbb{R}^{p \mid q}$. Then $T_{f, U}: U \times J_{A} \otimes V \rightarrow U \times J_{B} \otimes V$ is given by

$$
T_{f, U}\left(u, \sum_{i} a_{i} \otimes v_{i}\right)=\left(u, \sum_{i} f\left(a_{i}\right) v_{i}\right)
$$

for all $u \in_{S} U, a_{i} \in_{S} J_{A}, v_{i} \in_{S} V$.
If $f$ is surjective, let $W$ be a graded vector space complement of $\operatorname{ker} f$. Under the isomorphisms $B \times \operatorname{ker} f \rightarrow W \times \operatorname{ker} f \rightarrow V$, the map $f$ corresponds to $p_{1}: W \times \operatorname{ker} f \rightarrow B$.

If $f$ is injective, let $W$ be a graded vector space complement of $f(A)$. Under the isomorphisms $A \times$ coker $f \rightarrow f(A) \times W \rightarrow B$, the map $f$ corresponds to (id, 0) : $A \rightarrow A \times$ coker $f$.
(3). We observe that $T_{A}(\mathbb{R})=A$ as a linear supermanifold. The structure maps $+, \cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ give rise to $T_{A}(+), T_{A}(\cdot)$. By the computation in (2), these are the same as the addition and multiplication of $A$. Hence, any natural transformation $T_{A} \rightarrow T_{B}$ gives rise to an even linear map which commutes with $\cdot$. Naturality, applied to the embedding $\eta_{A}: \mathbb{R} \rightarrow A$, gives the unitality. But in view of Corollary 4.72 , this proves our claim.

### 4.5. The local structure of morphisms.

4.75. Derivatives can be given an expression in terms of coordinates; this approach is common in the literature.

Let $U$ be a superdomain in $\mathbb{R}^{p \mid q}$, with coordinates $(x, \xi)$ given by a basis of $\left(\mathbb{R}^{p \mid q}\right)^{*}$. Assume given a morphism $\varphi: U \rightarrow V$, where $V$ is some linear supermanifold. By Proposition 4.65, one may view $d \varphi$ as an element of

$$
\left(\Gamma\left(\mathcal{O}_{U}\right) \otimes \underline{\operatorname{Hom}}\left(\mathbb{R}^{p \mid q}, V\right)\right)_{0}=\operatorname{Hom}_{\Gamma\left(\mathcal{O}_{U}\right)}\left(\Gamma\left(\mathcal{O}_{U}\right) \otimes \mathbb{R}^{p \mid q}, \Gamma\left(\mathcal{O}_{U}\right) \otimes V\right) .
$$

Thus, if $x_{i}^{*}, \xi_{j}^{*}$ are the dual basis of $\mathbb{R}^{p \mid q}$, given by

$$
x_{i}\left(x_{j}^{*}\right)=\delta_{i j}, \xi_{i}\left(x_{j}^{*}\right)=0, x_{i}\left(\xi_{j}^{*}\right)=0, \xi_{i}\left(\xi_{j}^{*}\right)=\delta_{i j},
$$

then we may define

$$
\left(\frac{\partial}{\partial x_{i}} \varphi\right)=d \varphi\left(1 \otimes x_{i}^{*}\right) \quad \text { and } \quad\left(\frac{\partial}{\partial \xi_{i}} \varphi\right)(u)=d \varphi\left(1 \otimes \xi_{i}^{*}\right) .
$$

Since $\left(\Gamma\left(\mathcal{O}_{U}\right) \otimes V\right)_{0}=V(U)$, this determines morphisms $\frac{\partial}{\partial x_{i}} \varphi, \frac{\partial}{\partial \xi_{i}} \varphi: U \rightarrow V$.
If $V$ is endowed with the linear coordinate system $(y, \eta)$, then the matrix representation of $d \varphi$ in the corresponding bases has the entries $\frac{\partial \varphi^{*}\left(y_{i}\right)}{\partial x_{j}}, \frac{\partial \varphi^{*}\left(y_{i}\right)}{\partial \xi_{j}}$, $\frac{\partial \varphi^{*}\left(\eta_{i}\right)}{\partial x_{j}}$ and $\frac{\partial \varphi^{*}\left(\eta_{i}\right)}{\partial \xi_{j}}$. This follows from the chain rule. In shorthand, one writes

$$
d \varphi=\left(\begin{array}{ll}
\frac{\partial \varphi^{*}(y)}{\partial x} & \frac{\partial \varphi^{*}(y)}{\partial \xi} \\
\frac{\partial \varphi^{*}(\eta)}{\partial x} & \frac{\partial \varphi^{*}(\eta)}{\partial \xi}
\end{array}\right)
$$

This is called the Jacobian matrix of $\varphi$ in the coordinates $(x, \xi)$ and $(y, \eta)$.
For $V=\mathbb{R}^{1 \mid 1}$, the partial derivatives of superfunctions on $U$ are again superfunctions. An easy computation using the chain rule shows that $\frac{\partial}{\partial x_{i}}$ are even derivations, whereas $\frac{\partial}{\partial \xi_{i}}$ are odd derivations.

In terms of the partial derivatives, we may express $d \varphi$ as

$$
d \varphi=\sum_{i} \frac{\partial \varphi}{\partial x_{i}} \otimes x_{i}-\sum_{i}(-1)^{|\varphi|} \frac{\partial \varphi}{\partial \xi_{i}} \otimes \xi_{i}
$$

Moreover,

$$
\frac{\partial x_{i}}{\partial x_{j}}=\delta_{i j}, \frac{\partial \xi_{i}}{\partial \xi_{j}}=0, \frac{\partial x_{i}}{\partial \xi_{j}}=0, \frac{\partial \xi_{i}}{\partial \xi_{j}}=\delta_{i j}
$$

Definition 4.76. A morphism $\varphi: X \rightarrow Y$ is called a local isomorphism if there exist open covers $\left(U_{i}\right),\left(V_{i}\right)$ of $X$ and $Y$, respectively, such that $\varphi$ induces isomorphisms $U_{i} \rightarrow V_{i}$.

We have following version of the Inverse Function Theorem for superdomains.

Proposition 4.77. Let $\varphi: X \rightarrow Y$ be a morphism of superdomains $X \subset \mathbb{R}^{p \mid q}$, $Y \subset \mathbb{R}^{r \mid s}$. The following are equivalent:
(1). $\varphi_{0}\left(X_{0}\right)$ is open and $\varphi: X \rightarrow \varphi(X)$ is a local isomorphism,
 invertible, and
(3). for any $x \in X_{0}$, the value of $d \varphi$ at $x$ is invertible.

Proof. It is clear that $(1) \Rightarrow(2)$, from Proposition 4.59. Trivially, $(2) \Rightarrow(3)$.
Thus, assume (3). Then $p|q=r| s$. Let $(y, \eta)$ and $(x, \xi)$ be coordinate systems on $Y$ and $X$, respectively. Let $J_{o}$ denote the germ at $o \in X_{0}$ of the Jacobian matrix

$$
\left(\begin{array}{ll}
\frac{\partial \varphi^{*}(y)}{\partial x} & \frac{\partial \varphi^{*}(y)}{\partial \xi} \\
\frac{\partial \varphi^{*}(\eta)}{\partial x} & \frac{\partial \varphi^{*}(\eta)}{\partial \xi}
\end{array}\right) .
$$

Then

$$
j_{0}^{*}\left(J_{o}\right)=\left(\begin{array}{cc}
j_{0}^{*}\left(\frac{\partial \varphi^{*}(y)}{\partial x}\right) & 0 \\
0 & j_{0}^{*}\left(\frac{\partial \varphi^{*}(\eta)}{\partial \xi}\right)
\end{array}\right)
$$

is invertible by assumption. In particular, $j_{0}^{*}\left(\frac{\partial \varphi^{*}(y)}{\partial x}\right)=\frac{\partial \varphi_{0}\left(y_{0}\right)}{\partial x_{0}}$ is invertible. Hence, the classical inverse function theorem applies: possibly replacing $X$ and $Y$ by open subspaces, $\varphi_{0}: X_{0} \rightarrow Y_{0}$ is an isomorphism. Hence, we may assume that $X=Y$ and $\varphi_{0}=\mathrm{id}$.

Possibly shrinking $X$ further, we may assume that the $q \times q$ matrix $A=j_{0}^{*}\left(\frac{\partial \varphi^{*}(\eta)}{\partial \xi}\right)$ with entries in $\mathcal{C}^{\varpi}\left(X_{0}\right)$ is invertible. Let $\zeta=A \xi$. There is a unique algebra automorphism of $\mathcal{O}_{X}$ which is the identity on $\mathcal{C}_{X_{0}}^{\infty}$ and which maps $\xi_{i}$ to $\zeta_{i}$. Hence, we may assume that

$$
\varphi^{*}\left(y_{i}\right) \equiv x_{i} \quad(\bmod N) \quad \text { and } \quad \varphi^{*}\left(\eta_{i}\right) \equiv \xi_{i} \quad\left(\bmod N^{2}\right)
$$

where $N=\Gamma\left(\mathcal{N}_{X}\right)$.
Define $\psi_{k}: X \rightarrow X$ by requiring $\psi_{0}^{*}\left(x_{i}\right)=y_{i}, \psi_{0}^{*}\left(\xi_{i}\right)=\eta_{i}$, and setting $\psi_{k}=\left(\mathrm{id}, \psi_{k}^{*}\right)$ for $k \geqslant 1$, where

$$
\psi_{k}^{*}(f)=\psi_{0}^{*}\left(f-\varphi^{*} \psi_{k-1}^{*}(f)\right)+\psi_{k-1}^{*}(f) .
$$

Let $\Delta_{k}(f)=\varphi^{*} \psi_{k}^{*}(f)-f$. Then
$\Delta_{k+1}(f)=\varphi^{*} \psi_{k+1}^{*}(f)-f=-\varphi^{*} \psi_{0}^{*}\left(\Delta_{k}(f)\right)+\varphi^{*} \psi_{k}^{*}(f)-f=-\Delta_{0}\left(\Delta_{k}(f)\right)$.
By assumption, $\Delta_{0}\left(\mathcal{O}_{X}\right) \subset \mathcal{N}_{X}$, so $\Delta_{k}\left(\mathcal{O}_{X}\right) \subset \mathcal{N}_{X}^{k+1}$. In particular, $\psi_{k}^{*} \varphi^{*}=\operatorname{id}_{\mathcal{O}_{X}}$ for $k \geqslant q$. Thus, $\varphi$ has a left inverse $\psi=\psi_{q}$. Applying Proposition 4.59, we see that the assumption of (3) applies to $\psi$. By what we have proved, $\psi$ possesses a left inverse $\phi$, possibly after shrinking $X$. Then $\phi=\phi \psi \varphi=\varphi$ on $X$, and this proves the assertion.

To globalise the Inverse Function Theorem, we introduce the cotangent sheaf of a supermanifold.
Construction 4.78. Let $X$ be a supermanifold and $\varphi_{i}: U_{i} \rightarrow X$ open embeddings such that the open subspaces $V_{i} \varphi_{i}\left(U_{i}\right)$ cover $X$. For any index $i$, we have $T U_{i}=U_{i} \times \mathbb{R}^{p_{i} \mid q_{i}}$, and we are given the bundle projection $\pi$ on $T X$.

Define subsheaves $\mathcal{T}_{i}{ }^{*} \subset \pi_{0, *} \mathcal{O}_{T V_{i}}$ on $V_{i}$ as follows: Let $\mathcal{T}_{i}{ }^{*}\left(\pi_{0}^{-1}(U)\right)$ consist of those $f:\left.T V_{i}\right|_{U} \rightarrow \mathbb{R}^{1 \mid 1}$ such that $g=\left(T \varphi_{i}\right)^{*}(f)$ satisfies
$g(u, \lambda v+w)=\lambda g(u, v)+g(u, w)$ for all $u \in_{S} U, \lambda \in_{S} \mathbb{R}, v, w \in_{S} \mathbb{R}^{p_{i} \mid q_{i}}$. In other words, $g \in \mathcal{O}_{U_{i}}\left(\varphi_{i, 0}^{-1}(U)\right) \otimes\left(\mathbb{R}^{p_{i} \mid q_{i}}\right)^{*}$, by Proposition 4.65.

Consider the identity sheaf morphism $\pi_{0, *} \mathcal{O}_{T V_{j} \mid V_{i j}} \rightarrow \pi_{0, *} \mathcal{O}_{T V_{i} \mid V_{i j}}$. We claim that it restricts to a sheaf morphism $\psi_{i j}:\left.\left.\mathcal{T}_{j}^{*}\right|_{V_{i j}} \rightarrow \mathcal{T}_{i}{ }^{*}\right|_{V_{i j}}$. Indeed, let $f \in \mathcal{T}_{i}^{*}(U)$ where $U \subset V_{i j, 0}$ is open. Let $g_{i}=\left(T \varphi_{i}\right)^{*} f$ and $g_{j}=\left(T \varphi_{j}\right)^{*} f$. Then by Theorem 4.67,

$$
\begin{aligned}
g_{i}(u, v) & =\left(T \varphi_{i}\right)^{*} f(u, v)=\left(T\left(\varphi_{j}^{-1} \circ \varphi_{i}\right)\right)^{*}\left(T \varphi_{j}\right)^{*} f(u, v) \\
& =g_{j}\left(\varphi_{j}^{-1}\left(\varphi_{i}(u)\right), d\left(\varphi_{j}^{-1} \circ \varphi_{i}\right)(u) v\right) .
\end{aligned}
$$

The claim follows from Proposition 4.58.
By construction, we have $\psi_{i k} \circ \psi_{k j} \circ \psi_{j i}=$ id on $V_{i j k}$. Thus, by Proposition 3.18 , there exists an up to canonical isomorphism unique $\mathcal{O}_{X}$-module sheaf $\mathcal{T}_{X}^{*}$ on $X_{0}$ such that $\left.\mathcal{T}_{X}^{*}\right|_{V_{i}} \cong \mathcal{T}_{i}$; it is called the cotangent sheaf of $X$. By construction, this module sheaf is locally free of local rank $\operatorname{dim}_{x} X$ at $x$.

By the above considerations, any morphism $\varphi: X \rightarrow Y$ gives rise to a sheaf morphism $(T \varphi)^{*}: \mathcal{T}_{Y}^{*} \rightarrow \varphi_{0 *} \mathcal{T}_{X}^{*}$.

Definition 4.79. Let $X$ be a supermanifold. Denote by $\mathcal{T}_{X}$ the sheaf $\operatorname{Der}\left(\mathcal{O}_{X}\right)$ of graded derivations, that is

$$
\mathcal{T}_{X}(U)=\left\{\delta \in \underline{\operatorname{Hom}}\left(\left.\mathcal{O}_{X}\right|_{U}, \mathcal{O}_{X} \mid U\right) \mid \delta(f \cdot g)=\delta(f) \cdot g+(-1)^{|f||\delta|} f \cdot \delta(g)\right\}
$$

The sheaf $\mathcal{T}_{X}$ is a left $\mathcal{O}_{X}$-module. We call it the tangent sheaf of $X$.
For any morphism $\varphi: X \rightarrow Y$, there is a morphism of $\mathcal{O}_{X}$-module sheaves

$$
\mathcal{T} \varphi: \mathcal{T}_{X} \rightarrow \varphi^{*} \mathcal{T}_{Y}=\mathcal{O}_{X} \otimes_{\varphi_{0}^{-1} \mathcal{O}_{X}} \varphi_{0}^{-1} \mathcal{T}_{Y}
$$

given by

$$
\mathcal{T} \varphi(\delta)(f)=\delta\left(\varphi^{*}(f)\right) \quad \text { for all } \delta \in \mathcal{T}_{X}(U), f \in \mathcal{O}_{Y}(V), \varphi_{0}(U) \subset V
$$

Proposition 4.80. For any supermanifold $X$, we have a natural isomorphism $\phi: \underline{\mathcal{H o m}}_{\mathcal{O}_{X}}\left(\mathcal{T}_{X}^{*}, \mathcal{O}_{X}\right) \rightarrow \mathcal{T}_{X}$ of $\mathcal{O}_{X}$-moduless. Explicity, the naturality means

$$
\phi\left(h \circ(T \varphi)^{*}\right)=\mathcal{T} \varphi(\phi(h)) \quad \text { for all } \varphi: X \rightarrow Y .
$$

In particular, the $\mathcal{O}_{X}$-module $\mathcal{T}_{X}$ is locally free of local rank $\operatorname{dim}_{x} X$.
Lemma 4.81. Let $U$ be a superdomain with linear coordinate system $\left(e_{i}\right)$. For any $\delta \in \Gamma\left(\mathcal{T}_{U}\right)$ and $f \in \Gamma\left(\mathcal{O}_{U}\right)$, we have

$$
\delta(f)=\sum_{i} \delta\left(e_{i}\right) \frac{\partial}{\partial e_{i}} .
$$

Proof. W.l.o.g., $\delta$ is homogeneous. Then, set $\delta^{\prime}(f)=\delta(f)-\sum_{i} \delta\left(e_{i}\right) \frac{\partial f}{\partial e_{i}}$. In view of Proposition 4.11, we need to prove that $\delta^{\prime}(f)_{x} \in \mathfrak{m}_{x}^{q+1}$ for all $x \in U_{0}$.

We have $\delta^{\prime}(1)=0$ and $\delta^{\prime}\left(e_{i}\right)=0$, so $\delta^{\prime}(f)=\delta(f-p)$ for any polynomial $p$ in $e_{i}$. Fix $x \in U_{0}$. By Corollary 4.17, there exists $p$ such that $(f-p)_{x} \in \mathfrak{m}_{x}^{q+1}$.

By Proposition 4.16, $\mathfrak{m}_{x}$ is generated by $u_{i}=e_{i}-e_{i}(x)$. Now,

$$
\delta^{\prime}\left(u_{i} \cdot g\right)=\delta\left(u_{i}\right)^{\prime} \cdot g+(-1)^{\left|u_{i}\right|\left|\delta^{\prime}\right|} u_{i} \cdot \delta^{\prime}(g)=(-1)^{\left|u_{i}\right|\left|\delta^{\prime}\right|} u_{i} \cdot \delta^{\prime}(g) .
$$

It follows that $\delta_{x}^{\prime}\left(\mathfrak{m}_{x}\right) \subset \mathfrak{m}_{x}$ and hence, that $\delta_{x}^{\prime}\left(\mathfrak{m}_{x}^{q+1}\right) \subset \mathfrak{m}_{x}^{q+1}$. This proves the assertion.
Proof of Proposition 4.80. Let $U$ be an open subspace of $V=\mathbb{R}^{p \mid q}$. Recall $\mathcal{T}_{U}=\mathcal{O}_{U} \otimes V^{*}$. In view of Proposition 4.65, we may define $\phi=\phi_{U}$ : $\mathcal{H o m}_{\mathcal{O}_{U}}\left(\mathcal{T}_{U}^{*}, \mathcal{O}_{U}\right) \rightarrow \mathcal{T}_{U}$ by

$$
\phi(h)(f)=h(d f) \quad \text { for all } h \in{\underset{\mathcal{H}}{ }\left(m_{\mathcal{O}_{U}}\right.}\left(\mathcal{T}_{U}^{*}, \mathcal{O}_{U}\right)(W), f \in \mathcal{O}_{U}(W)
$$

This clearly defines superderivations of the correct degree.
To see that this defines a natural morphism of sheaves, it suffices to check this fact for superdomains, by Proposition 3.18 and Proposition 4.4. Thus, let $\varphi: U \rightarrow U^{\prime}$ be a morphism of superdomains. Then

$$
\begin{aligned}
\phi_{U^{\prime}}\left(h \circ\left(T \varphi^{*}\right)\right)(f) & =h\left((T \varphi)^{*} d f\right)=h(d f(\varphi(u))(d \varphi(u) v)) \\
& =h(d(f \circ \varphi))=\phi_{U}(h)\left(\varphi^{*} f\right)=(\mathcal{T} \varphi)\left(\phi_{U}(h)\right)(f)
\end{aligned}
$$

for $u=\operatorname{id}_{U} \epsilon_{U} U, v=\operatorname{id}_{V} \in_{V} V$ the generic points. Here, we have used Proposition 4.59 and Theorem 4.67. This shows the naturality.

To see that $\phi$ is an isomorphism, it is sufficient to do this the case of a superdomain. Given a superderivation $\delta$, define $h=\psi(\delta)$ by

$$
h\left(\sum_{i} f_{i} \otimes \mu_{i}\right)=\sum_{i}(-1)^{\left|f_{i}\right||\delta|} f_{i} \delta\left(\mu_{i}\right) .
$$

It is straightforward to check that this defines an $\mathcal{O}_{U}$-linear map.
For any $\mu \in V^{*}$, we have $d \mu=1 \otimes \mu$. Hence,

$$
\psi(\phi(h))(f \otimes \mu)=(-1)^{|f||h|} f \phi(h)(\mu)=(-1)^{|f||h|} f \cdot h(1 \otimes \mu)=h(f \otimes \mu),
$$

so $\psi \circ \phi=$ id and $\phi$ is injective.
Conversely,

$$
\begin{aligned}
\phi(\psi(\delta))(f) & =\psi(\delta)(d f)=\sum_{i}(-1)^{\left(\left|e_{i}\right|+|f|\right)\left(|\delta|+\left|e_{i}\right|\right)} \frac{\partial f}{\partial e_{i}} \delta\left(e_{i}\right) \\
& =\sum_{i} \delta\left(e_{i}\right) \frac{\partial f}{\partial e_{i}}=\delta(f)
\end{aligned}
$$

so that $\phi \circ \psi=\mathrm{id}$, and the assertion is proven.
Corollary 4.82. Let $X$ be a supermanifold and $x \in X_{0}$. We have

$$
\mathcal{T}_{X, x} \cong \underline{\operatorname{Der}}\left(\mathcal{O}_{X, x}, \mathbb{R}\right) \cong \underline{\operatorname{Hom}}\left(\mathfrak{m}_{X, x} / \mathfrak{m}_{X, x}^{2}, \mathbb{R}\right) \quad \text { and } \quad \mathcal{T}_{X, x}^{*} \cong \mathfrak{m}_{x} / \mathfrak{m}_{x}^{2}
$$

If $\varphi: X \rightarrow Y$ is a morphism, then the sheaf map $\mathcal{T} \varphi$ induces an even linear $\operatorname{map} T_{x} \varphi=(\mathcal{T} \varphi)_{x}: \mathcal{T}_{X, x} \rightarrow \mathcal{T}_{Y, \varphi_{0}(x)}$.
Definition 4.83. The super-vector space $T_{x} X=\mathcal{T}_{X, x}$ is called the tangent space of $X$ at $x$. The map $T_{x} \varphi: T_{x} X \rightarrow T_{\varphi_{0}(x)} Y$ is called the tangent map of $\varphi$ at $x$.

A morphism $\varphi: X \rightarrow Y$ is called a submersion if $T_{x} \varphi$ is surjective for all $x \in X_{0}$. It is called a surjective submersion if in addition, $\varphi_{0}$ is surjective. Furthermore, $\varphi$ is called an immersion if $T_{x} \varphi$ is surjective for all $x \in X_{0}$. It is called an injective immersion if in addition, $\varphi_{0}$ is injective. An injective immersion is called an embedding if $\varphi_{0}: X_{0} \rightarrow Y_{0}$ is a topological embedding, that is, its corestriction $X_{0} \rightarrow \varphi_{0}\left(X_{0}\right)$ is a homeomorphism if $\varphi_{0}\left(X_{0}\right)$ is endowed with the relative topology of $Y_{0}$.

Theorem 4.84. Let $\varphi: X \rightarrow Y$ be a morphism of supermanifolds which is at the same time an immersion and a submersion. Then $\varphi_{0}\left(X_{0}\right)$ is open, so that $\varphi(X)$ exists, and $\varphi: X \rightarrow \varphi(X)$ is a local isomorphism.

Proof. The problem is local, and the statement will therefore follow once we are able to express $T_{x} \varphi$ in terms of the Jacobian.

Thus, we may assume that $X \subset V$ and $Y \subset W$ are open subspaces of linear supermanifolds. Let $x \in X_{0}$. We have, for all $\delta \in T_{x} X, f \in \mathcal{O}_{Y, \varphi_{0}(x)}$,

$$
T_{x} \varphi(\delta)(f)=\delta\left(\varphi^{*}(f)\right)_{x}
$$

In particular, if $\left(e_{i}\right)$ is a coordinate system on $V$, and $\left(f_{i}\right)$ is a coordinate system on $W$, then

$$
T_{x} \varphi\left(\frac{\partial}{\partial e_{i}}\right)\left(f_{i}\right)=\frac{\partial \varphi^{*}\left(f_{i}\right)}{\partial e_{i}}
$$

so in view of 4.75 , the matrix expression of the germ of $d \varphi$ at $x$ corresponds to the linear map $T_{x} \varphi$.

Corollary 4.85. Let $\varphi: X \rightarrow Y$ be a morphism of supermanifolds such that $\varphi_{0}: X_{0} \rightarrow Y_{0}$ is bijective and $\varphi$ is an immersion and a submersion.

Proof. By Theorem 4.84, $\varphi: X \rightarrow Y$ is a local isomorphism. In particular, $\varphi_{x}^{*}: \mathcal{O}_{Y, \varphi_{0}(x)} \rightarrow \mathcal{O}_{X, x}$ is an isomorphism for each $x \in X_{0}$, and $\varphi_{0}: X_{0} \rightarrow Y_{0}$ is a homeomorphism. In view of Corollary 3.9, $\varphi^{*}: \mathcal{O}_{Y} \rightarrow \varphi_{0, *} O_{X}$ is an isomorphism of sheaves on $Y_{0}=\varphi_{0}\left(X_{0}\right)$. Hence the assertion.

Proposition 4.86. Let $\varphi: X \rightarrow Y$ be a morphism and $x \in X_{0}$ such that $T_{x} \varphi: T_{x} X \rightarrow T_{y} Y$ is surjective, where $y=\varphi_{0}(x)$. Then there exists an open neighbourhood $U \subset Y$ of $y$ and a supermanifold $Z$ such that $\varphi:\left.X\right|_{U} \rightarrow U$ is isomorphic as a relative supermanifold to $p_{1}: U \times Z \rightarrow U$.

Proof. The problem is local, so that we may assume that $X$ and $Y$ are superdomains and $\varphi$ is a submersion.

Choose linear coordinate systems $\left(e_{i}\right)$ of $X$ and $\left(f_{i}\right)$ of $Y$ such that $\left(\frac{\partial \varphi^{*}\left(f_{i}\right)}{\partial e_{j}}\right)_{1 \leqslant i, j \leqslant p+q}$ is invertible, where $p \mid q=\operatorname{dim} Y$. Passing to an open subspace, we may assume $X=Y^{\prime} \times Z$ for some superdomains, and so the morphism $\left(\varphi, p_{2}\right): X \rightarrow Y \times Z$ is defined.

It satisfies the assumption of Theorem 4.84, so that passing to subdomains, we have an inverse $\psi$. Then $p_{1}=\varphi \circ \psi$, by construction.

Proposition 4.87. Let $\varphi: X \rightarrow Y$ be a morphism and $x \in X_{0}$ such that $T_{x} \varphi: T_{x} X \rightarrow T_{y} Y$ is injective, where $y=\varphi_{0}(x)$. Then there exist open neighbourhoods $U \subset X$ of $x, V \subset Y$ of $y$ such that $\varphi^{-1}(V) \supset U$, a supermanifold $Z$, and a point $z \in Z_{0}$, such that $\varphi: U \rightarrow V$ is isomorphic to (id, $z$ ) : $U \rightarrow U \times Z$.

Proof. Again, we may assume that $X$ and $Y$ are superdomains and $\varphi$ is an immersion. Choose linear coordinate systems $\left(e_{i}\right)$ of $X$ and $\left(f_{i}\right)$ of $Y$ such that $\left(\frac{\partial \varphi^{*}\left(f_{i}\right)}{\partial e_{j}}\right)_{1 \leqslant i, j \leqslant p+q}$ is invertible, where $p \mid q=\operatorname{dim} X$.

Let $m \mid n=\operatorname{dim} Y$, and let $Z \subset \mathbb{R}^{m-p \mid n-q}$ be a neighbourhood of 0 , with linear coordinates $e_{i}, p+q<i \leqslant m+n$, of parity $\left|e_{i}\right|=\left|f_{i}\right|$. Define $\psi: X \times Z \rightarrow Y$ by

$$
\psi^{*}\left(f_{i}\right)= \begin{cases}p_{1}^{*} \varphi^{*}\left(f_{i}\right) & i \leqslant p+q \\ p_{1}^{*} \varphi^{*}\left(f_{i}\right)+p_{2}^{*}\left(e_{i}\right) & i>p+q\end{cases}
$$

Then the Jacobian matrix of $\psi$ takes the form

$$
\left(\begin{array}{cc}
\left(\frac{\partial \varphi^{*}\left(f_{k}\right)}{\partial e_{\ell}}\right)_{1 \leqslant k, \ell \leqslant p+q} & 0 \\
* & 1
\end{array}\right)
$$

so that $\psi$ satisfies the assumptions of Theorem 4.84, and possesses a local inverse. Let $\phi=\psi^{-1} \circ \varphi$. Observe that $\phi^{*}\left(p_{1}^{*}\left(f_{i}\right)\right)=\varphi^{*}\left(f_{i}\right)$ for $i \leqslant p+q$, so that $p_{1} \circ \phi=\mathrm{id}$. It follows that for $i>p+q$, one has

$$
\begin{aligned}
\phi^{*}\left(p_{2}^{*}\left(e_{i}\right)\right) & =-\phi^{*}\left(p_{1}^{*} \varphi^{*}\left(f_{i}\right)\right)+\phi^{*}\left(p_{1} *\left(\varphi^{*}\left(f_{i}\right)\right)+p_{2}^{*}\left(e_{i}\right)\right) \\
& =-\varphi^{*}\left(f_{i}\right)+\varphi^{*}\left(f_{i}\right)=0
\end{aligned}
$$

Hence, $\phi=(\mathrm{id}, 0): X \rightarrow X \times Z$.
Proposition 4.88. Let $\varphi: X \rightarrow Y$ be a morphism such that $\varphi_{0}\left(X_{0}\right)$ is locally closed in $Y_{0}$, and $x \in X_{0}$ such that $T_{x} \varphi: T_{x} X \rightarrow T_{y} Y$ is injective, where $y=\varphi_{0}(x)$. Then there an exists open neighbourhood $V \subset Y$ of $y$, a supermanifold $Z$, and a point $z \in Z_{0}$, such that $\varphi: \varphi^{-1}(V) \rightarrow V$ is isomorphic to id $\times z: \varphi^{-1}(V) \rightarrow \varphi^{-1}(V) \times Z$.

Proof. The set $\varphi_{0}\left(X_{0}\right)$ is closed in an open neighbourhood, so we may assume that it is closed. If $y^{\prime} \notin \varphi_{0}\left(X_{0}\right)$, then there is an open neighbourhood $W$ of
$y^{\prime}$ such that $\varphi_{0}^{-1}(W)=\varnothing$. Hence, in the situation of Proposition 4.87, we may construct $V$ such that $U=\varphi^{-1}(V)$.

Corollary 4.89. An injective immersion $\varphi: X \rightarrow Y$ is an embedding if and only if $\varphi_{0}\left(X_{0}\right)$ is locally closed.

Proof. If $\varphi$ is an embedding, then by assumption, $\varphi_{0}\left(X_{0}\right)$ is locally compact, hence locally closed. Conversely, Proposition 4.88 shows that $\varphi_{0}$ possesses local continuous inverses on $\varphi_{0}\left(X_{0}\right)$, endowed with the relative topology. Since $\varphi_{0}$ is injective, it is a homeomorphism.

Corollary 4.90. Let $\varphi: X \rightarrow Y$ be an embedding and $f \in \varphi_{0 *} \mathcal{O}_{X}(U)$, where $U \subset Y_{0}$ is open. For each $y \in U \cap \varphi_{0}\left(X_{0}\right)$, there exists a neighbourhood $V \subset U$ of $y$ and $h \in \mathcal{O}_{Y}(V)$ such that $\varphi^{*}(h)=\left.f\right|_{V}$.

Proof. In view of Proposition 4.88 and Corollary 4.89, we may assume that $\varphi=(\operatorname{id}, 0): X \rightarrow X \times Z=Y$ where $X, Y, Z$ are open neighbourhoods of 0 , and that $y=(0,0)$. Given $f$, define $h$ on generalised points by $h(x, z)=f(x)$.

Definition 4.91. Let $X$ be a supermanifold and $Y_{0} \subset X_{0}$ a subspace. A subspace of $X$ on $Y_{0}$ is a super-ringed space $Y=\left(Y_{0}, \mathcal{O}_{Y}\right)$, together with a morphism $j_{Y}=\left(j_{Y, 0}, j_{Y}^{*}\right): Y \rightarrow X$ where $j_{Y, 0}: Y_{0} \rightarrow X_{0}$ is the inclusion and $j_{Y}^{*}: j_{Y, 0}^{-1} \mathcal{O}_{X} \rightarrow \mathcal{O}_{Y}$ is an epimorphism. If, moreover, $Y$ is a supermanifold, then it is called a subsupermanifold.

Any subspace $Y$ comes with a canonical ideal sheaf $\mathcal{I}_{Y}$. It is the subsheaf of $j_{Y, 0}^{-1} \mathcal{O}_{X}$ defined as the kernel of $j_{Y}^{*}$. The data of $Y_{0}$ and $\mathcal{I}_{Y}$ determine $Y$ uniquely up to isomorphism. Therefore, $\mathcal{I}_{Y}$ is called the defining ideal of $Y$.

Proposition 4.92. Let $\varphi: X \rightarrow Y$ be an injective immersion. The set $\varphi_{0}\left(X_{0}\right)$ carries a unique structure of supermanifold, denoted $\varphi(X)$, such that $\varphi$ factors into an isomorphism $X \rightarrow \varphi(X)$ and an injective immersion $\varphi(X) \rightarrow Y$. The supermanifold $\varphi(X)$ is a subsupermanifold of $Y$ if and only if $\varphi$ is an embedding, if and only if $\varphi_{0}\left(X_{0}\right)$ is locally closed. In this case, the defining ideal is $\mathcal{I}_{\varphi(X)}=\operatorname{ker}\left(\varphi^{*}: \mathcal{O}_{Y} \rightarrow \varphi_{0 *} \mathcal{O}_{X}\right)$.

Proof. Equip $\varphi_{0}\left(X_{0}\right)$ with the final topology with respect to $\varphi_{0}$. Then $\varphi_{0}$ induces a homeomorphism $\tilde{\varphi}_{0}: X_{0} \rightarrow \varphi_{0}\left(X_{0}\right)$. Defining $\mathcal{O}_{\varphi(X)}=\tilde{\varphi}_{0 *} \mathcal{O}_{X}$, we obtain a super-ringed space $\varphi(X)$. There manifestly is an isomorphism $\tilde{\varphi}: X \rightarrow \varphi(X)$, so the latter is indeed a supermanifold. Moreover, $\varphi \circ \tilde{\varphi}^{-1}$ is an injective immersion $\varphi(X) \rightarrow Y$. The uniqueness of the supermanifold structure (up to isomorphism) is trivial.

Let $j_{\varphi_{0}\left(X_{0}\right), 0}: \varphi_{0}\left(X_{0}\right) \rightarrow Y_{0}$ be the inclusion. By definition, $\varphi(X)$ is a subsupermanifold if and only if $\varphi_{0}\left(X_{0}\right)$ carries the relative topology of $Y_{0}$, and $\varphi^{*}$ induces an epimorphism

$$
\mathcal{O}_{Y} \rightarrow j_{\varphi_{0}\left(X_{0}\right), 0 *} \mathcal{O}_{\varphi(X)}=\varphi_{0 *} \mathcal{O}_{X}
$$

In consequence of Corollary 4.89, the condition on the topology of $\varphi_{0}\left(X_{0}\right)$ is satisfied if and only if $\varphi$ is an embedding, if and only if $\varphi_{0}\left(X_{0}\right)$ is locally closed. Moreover, if $\varphi$ is an embedding, then Corollary 4.90 and Corollary 3.9 imply that $\varphi^{*}$ indeed induces an epimorphism of sheaves.

Definition 4.93. If $\varphi: X \rightarrow Z$ and $\psi: Y \rightarrow Z$ are morphisms, then a supermanifold $W$ is called the fibre product of $\varphi$ and $\psi$, and denoted $X \times_{Z} Y$, if it is the product of $\varphi$ and $\psi$ in the category of morphisms $V \rightarrow Z$. That is, on the level of $S$-points,

$$
\left(X \times_{Z} Y\right)(S)=\{(x, y) \in X(S) \times Y(S) \mid \varphi(x)=\psi(y)\}
$$

This determines $X \times{ }_{Z} Y$ uniquely up to isomorphism (by the Yoneda lemma).
Proposition 4.94. Let $X, Y, Z$ be supermanifold, $\varphi: X \rightarrow Z$ be a surjective submersion and $\psi: Y \rightarrow Z$ any morphism. Then $X \times_{Z} Y$ exists, is a closed subsupermanifold of $X \times Y$, and $p_{2}: X \times{ }_{Z} Y \rightarrow Y$ is a surjective submersion. Its dimension is

$$
\operatorname{dim}_{(x, y)}\left(X \times_{Z} Y\right)=\operatorname{dim}_{x} X-\operatorname{dim}_{\varphi_{0}(x)} Z+\operatorname{dim}_{y} Y
$$

The defining ideal $\mathcal{I}_{X \times{ }_{Z} Y}$ of $X \times_{Z} Y$ as a subspace of $X \times Y$ is the image under $(\varphi \times \psi)^{*}$ of $\mathcal{I}_{\Delta_{Z}}$, the defining ideal of the diagonal $\Delta_{Z} \subset Z \times Z$.

In particular, if $\psi=j_{Y}: Y \rightarrow Z$ is the inclusion of an (open/closed) subsupermanifold, then the subspace $\varphi^{-1}(Y)=X \times_{Z} Y$ is an (open/closed) subsupermanifold of $X$, of codimension

$$
\operatorname{codim}_{X, x} \varphi^{-1}(Y)=\operatorname{codim}_{Z, \varphi_{0}(x)} Y
$$

Proof. To show that $X \times_{Z} Y$ exists and that the projection $X \times_{Z} Y \rightarrow Y$ is a surjective submersion, we may in view of Proposition 4.48 and Proposition 4.86 assume $X=X^{\prime} \times Z$ and $\varphi=p_{2}$. Then satisfies the universal property of $X \times_{Z} Y$, since we may compute

$$
X^{\prime}(S) \times Y(S) \cong\left\{(x, z, y) \in X^{\prime}(S) \times Z(S) \times Y(S) \mid z=\psi(y)\right\}
$$

where the natural bijection sends $(x, y)$ to $(x, \psi(y), y)$. Moreover, the morphism $p_{2}: X^{\prime} \times Y \rightarrow Y$ is certainly a surjective submersion.

Thus, $X \times_{Z} Y$ exists and $p_{2}: X \times_{Z} Y \rightarrow Y$ is a surjective submersion. We have the natural morphism $j: X \times{ }_{Z} Y \rightarrow X \times Y$ defined on generalised points by $j(x, y)=(x, y)$. Furthermore, by construction, $j$ locally identifies with morphism $X^{\prime} \times Y \rightarrow X^{\prime} \times Z \times Y$ given on points by $(x, y) \mapsto(x, \psi(y), y)$. This morphism is manifestly an injective immersion; since

$$
j_{0}\left(X_{0} \times_{Z_{0}} Y_{0}\right)=\left\{(x, y) \in X_{0} \times Y_{0} \mid \varphi_{0}(x)=\psi_{0}(y)\right\}
$$

is closed, so that $j$ is a closed embedding, and by Proposition 4.92, it identifies $X \times{ }_{Z} Y$ with a closed subsupermanifold of $X \times Y$.

If $\psi=j_{Y}$, then we have the embedding id $\times j_{Y}: X \times Y \rightarrow X \times Z$. The morphism (id, $\varphi$ ): $X \times X \times Z$ is also an embedding, and since

$$
(\mathrm{id}, \varphi)_{S}((X \times Z)(S))=\left\{(x, z) \in_{S} X \times Z \mid \varphi(x)=z\right\}
$$

we have that $(\mathrm{id}, \varphi)$ restricts to an isomorphism $\varphi^{-1}(Y) \rightarrow X \times{ }_{Z} Y$. Hence, $\varphi^{-1}(Y)$ is a supermanifold, and because $(\mathrm{id}, \varphi) \circ j_{\varphi^{-1}(Y)}$ is an embedding, so is $j_{\varphi^{-1}(Y)}$. Thus, $\varphi^{-1}(Y)$ is a subsupermanifold of $X$. If $Y_{0}$ is open resp. closed in $Z_{0}$, then $\varphi^{-1}\left(Y_{0}\right)$ is open resp. closed in $X_{0}$.

## 5. Quotients and actions

### 5.1. Godement's theorem on quotients.

Definition 5.1. Let $X$ be a supermanifold $R \rightarrow X \times X$ a subspace. Then $R$ is called an equivalence relation if $R(S)$ is an equivalence relation on $X(S)$ for any supermanifold $S$. We let $\pi_{j}=p_{j} \circ j_{R}: R \rightarrow X, j=1,2$.

Define the ringed space $X / R$ by letting $(X / R)_{0}=X_{0} / R_{0}$, with the quotient topology, and taking the structure sheaf $\mathcal{O}_{X / R}$ to be defined by

$$
\mathcal{O}_{X / R}(U)=\left\{f \in \mathcal{O}_{X}\left(\pi_{0}^{-1}(U)\right) \mid \pi_{1}^{*} f=\pi_{2}^{*} f\right\}
$$

where $\pi_{0}: X_{0} \rightarrow X_{0} / R_{0}$ is the canonical projection.
There is a canonical morphism $\pi: X \rightarrow X / R$, called the canonical projection, defined by $\pi=\left(\pi_{0}, \pi^{*}\right)$ where $\pi^{*}(f)=f$.

Remark 5.2. The quotient $X / R$ enjoys by definition the following universal property: If $\varphi: X \rightarrow Y$ is a morphism such that $\varphi \circ \pi_{1}=\varphi \circ \pi_{2}: R \rightarrow Y$, then there is a unique morphism $\tilde{\varphi}: X / R \rightarrow Y$ such that $\varphi=\tilde{\varphi} \circ \pi$.

Indeed, if $\varphi$ is given, then $\pi_{1}^{*} \varphi^{*}(f)=\pi_{2}^{*} \varphi^{*}(f)$ for any local section $f$ of $\mathcal{O}_{Y}$. Thus, $\varphi^{*}$ factors through $\pi^{*}: \pi_{0}^{-1} \mathcal{O}_{X / R} \rightarrow \mathcal{O}_{X}$.

The universal property may be rephrased as follows: If for any $(u, v) \in_{S} R$, one has $\varphi(u)=\varphi(v)$, then there is a unique morphism $\tilde{\varphi}: X / R \rightarrow Y$ such that $\tilde{\varphi} \circ \pi=\varphi$. Observe also that $Y$ may be any super-ringed space!

Theorem 5.3. Let $R$ be an equivalence relation on a supermanifold $X$. The following are equivalent.
(1). $X / R$ is a possibly non-paracompact supermanifold and $\pi: X \rightarrow X / R$ is a submersion.
(2). $R$ is a closed subsupermanifold of $X \times X$, and $\pi_{2}: R \rightarrow X$ is a submersion.

In this case, $\pi, \pi_{1}$, and $\pi_{2}$ are surjective submersions, and

$$
R \cong X \times_{X / R} X
$$

If in addition, $X_{0}$ is second countable or $\pi_{0}$ has the path-lifting property, then $X_{0} / R_{0}$ is paracompact.

Proof. (1) $\Rightarrow(2)$. By assumption, Proposition 4.94 implies that $X \times_{X / R} X$ is a closed subsupermanifold of $X \times X$. On points,

$$
\left(X \times_{X / R} X\right)(S)=\left\{(x, y) \in_{S} X \times X \mid \pi(x)=\pi(y)\right\}
$$

and in particular, $R_{0}=X_{0} \times_{X_{0} / R_{0}} X_{0}$ is locally closed in $X_{0} \times X_{0}$.
If $\varphi \in_{S} R$, then for $j_{R}(\varphi)=(x, y)$, one has

$$
x^{*}\left(\pi^{*}(f)\right)=\varphi^{*} j_{R}^{*} p_{1}^{*}(f)=\varphi^{*} \pi_{1}^{*}(f)=\varphi^{*} \pi_{2}^{*}(f)=\varphi^{*} j_{R}^{*} p_{2}^{*}(f)=y^{*}\left(\pi^{*}(f)\right)
$$

for all local sections $f$ of $\mathcal{O}_{X / R}$, so that $\pi(x)=\pi(y)$. In other words, $R(S) \subset\left(X \times_{X / R} X\right)(S)$, and there is a monomorphism $R \rightarrow X \times_{X / R} X$. In particular, $\mathcal{I}_{R}$ and $\mathcal{I}_{X \times X / R} X$ are both sheaves on $R_{0}$, and the latter is a subsheaf of the former.

By the very definition of $\mathcal{O}_{X / R}$, we have an equaliser diagram of sheaf morphisms on $X_{0}$,

$$
\pi_{0}^{-1} \mathcal{O}_{X / R} \xrightarrow{\pi^{*}} \mathcal{O}_{X} \xlongequal[\pi_{2}^{*}]{\pi_{1}^{*}} \pi_{j, 0 *} \mathcal{O}_{R}
$$

In other words, the kernel of $j_{R}^{*}=\left(\pi_{1} \times p_{2}\right)^{*}: \mathcal{O}_{X \times X} \rightarrow j_{R, 0 *} \mathcal{O}_{R}$ is exactly the image under $(\pi \times \pi)^{*}$ of $\operatorname{ker} \delta^{*}: \mathcal{O}_{X / R \times X / R} \rightarrow \delta_{0 *} \mathcal{O}_{X / R}$. The former is $\mathcal{I}_{R}$, and the latter is $\mathcal{I}_{X \times_{X / R} X}$. Thus, $R \cong X \times_{X / R} X$, which gives our claim.
$(2) \Rightarrow(1)$. By assumption, the $R_{0}$-saturation $\pi_{0}^{-1}\left(\pi_{0}(U)\right)=p_{2}\left(p_{1}^{-1}(U)\right)$ of $U$ is open for any open $U \subset X_{0}$ ( $p_{2}$ being a surjective submersion). Moreover, if $x, y \in X_{0}$ are inequivalent, there exists an open neighbourhood $U \subset X_{0} \times X_{0}$ of $(x, y)$ such that $U \cap R_{0}=\varnothing$, since $R_{0}$ is closed. Then there exist $V, W \subset X_{0}$, open neighbourhoods of $x$ and $y$, respectively, such that $V \times W \subset U$. By construction, $\pi_{0}(V) \cap \pi_{0}(W)=\varnothing$, and these sets are open. Thus, $X_{0} / R_{0}$ is a Hausdorff space and $\pi_{0}: X_{0} \rightarrow X_{0} / R_{0}$ is open.

If $X_{0}$ is second-countable, then so is $X_{0} / R_{0}$, since $\pi_{0}$ is open; in this case, $X_{0} / R_{0}$ is paracompact. Thus, it will be sufficient to prove that $X_{0} / R_{0}$ is paracompact if $\pi_{0}$ has the path-lifting property. It will follow from our considerations below that $X_{0} / R_{0}$ is locally Euclidean and $\pi_{0}: X_{0} \rightarrow X_{0} / R_{0}$ is a topological submersion. Hence, in view of Lemma ??, it will thus be sufficient to prove that every connected component is second countable. If $C$ is such a connected component, $\pi_{0}^{-1}(C)$ is closed in $X_{0}$ and therefore paracompact. If $C^{\prime}$ is a connected component of $\pi_{0}^{-1}(C)$, then it is second countable, by the lemma. Since $X_{0} / R_{0}$ is locally path-connected, $C$ is a path component. Since $\pi_{0}$ has the path-lifting property, we have $C=\pi_{0}\left(C^{\prime}\right)$, and since $C^{\prime}$ is second-countable, it follows that so is $C$. By Theorem ??, $C$ is paracompact. Since $C$ was arbitrary, so is $X$.

To prove that $X / R$ is a supermanifold and $\pi$ is a submersion is now a local problem. Let $x \in X_{0}$. Then $\pi_{2}^{-1}(x)$ is a closed subsupermanifold of $R$, by Proposition 4.94. It may be viewed as a subsupermanifold $R(x)$ of $X$ via the identification on points given by

$$
\begin{aligned}
\pi_{2}^{-1}(x)(S) & =\left\{(y, x) \in_{S} X \times X \mid(y, x) \in_{S} R\right\} \\
& \cong\left\{y \in_{S} X \mid(y, x) \in_{S} R\right\}=R(x)(S)
\end{aligned}
$$

By Proposition 4.88, there is an open neighbourhood $U \subset X$ of $x$ and a closed subsupermanifold $Z \subset U$ such that $T_{x} Z \oplus T_{x} R(x)=T_{x} X$. Then $\pi_{2}^{-1}(Z)$ is a closed subsupermanifold of the open subspace $\pi_{2}^{-1}(U) \subset R$. We claim that $\pi_{1}: \pi_{2}^{-1}(Z) \rightarrow U$ is a local isomorphism around $(x, x)$. We have

$$
\pi_{1} \circ \delta=\operatorname{id} \text { on } Z \text { and } \pi_{1} \circ(\mathrm{id}, x)=\mathrm{id} \text { on } R(x)
$$

(where $\delta$ is the diagonal morphism). Hence, we have

$$
T_{x} X=T_{x} Z \oplus T_{x} R(x) \subset \operatorname{im} T_{(x, x)} \pi_{1}
$$

On the other hand,

$$
\begin{aligned}
\operatorname{dim}_{(x, x)} \pi_{2}^{-1}(Z) & =\operatorname{dim} R_{(x, x)}-\operatorname{codim}_{X, x} Z \\
& =\operatorname{dim} R_{(x, x)}-\operatorname{dim}_{(x, x)} \pi_{2}^{-1}(x)=\operatorname{dim}_{x} X
\end{aligned}
$$

where the last equality follows since $\pi_{2}: R \rightarrow X$ is a submersion. Theorem 4.84 applies, and by shrinking $U$, we may assume that $\pi_{1}: \pi_{2}^{-1}(Z) \rightarrow U$ is an isomorphism. Let $\psi=\pi_{2} \circ \pi_{1}^{-1}: U \rightarrow Z$, so that $\pi_{1}^{-1}(u)=(u, \psi(u))$. By definition, $\psi(u) \in_{S} Z$ is uniquely determined by $(u, \psi(u)) \in_{S} R$.

By the universal property of $\pi$, there exists a unique morphism $\tilde{\psi}$ making the following diagram commutative:


On the other hand, let $\pi^{\prime}: Z \rightarrow U / R$ be defined by $\pi^{\prime}=\pi \circ j_{Z}$. We have $\psi \circ j_{Z}=\operatorname{id}_{Z}$, so

$$
\tilde{\psi} \circ \pi^{\prime}=\tilde{\psi} \circ \pi \circ j_{Z}=\psi \circ j_{Z}=\operatorname{id}_{Z}
$$

Conversely, we have $\left(j_{Z}(\psi(u)), u\right) \in_{S} R$ for all $u \in_{S} R$, by definition. This implies $\pi\left(j_{Z}(\psi(u))\right)=\pi(u)$. Since we may take $u=\operatorname{id}_{U} \in_{U} U$, this implies $\pi \circ j_{Z} \circ \psi=\pi$. (Yoneda's lemma does not apply directly, because we have as yet not shown that $U / R$ is a supermanifold.) Thus,

$$
\pi^{\prime} \circ \tilde{\psi} \circ \pi=\pi^{\prime} \circ \psi=\pi \circ j_{Z} \circ \psi=\pi
$$

The uniqueness statement in Remark 5.2 gives $\pi^{\prime} \circ \tilde{\psi}=\operatorname{id}_{U / R}$, and this implies that $U / R \cong Z$, so the former is indeed a supermanifold. Since $\psi$ is a surjective submersion, so is $\pi: U \rightarrow U / R$.

In the course of the proof, we have used the following fact.
Theorem 5.4. Let $X$ be a topological manifold, i.e. a Hausdorff space in which each point has an open beighbourhood homeomorphic to an open subset of some $\mathbb{R}^{j}$. The following are equivalent:
(1). $X$ is metrisable;
(2). $X$ is paracompact;
(3). the connected components of $X$ are $\sigma$-compact;
(4). the connected components of $X$ are second countable.

Proof. For the proof of the equivalence, w.l.o.g. we may assume that $X$ is connected.
$(1) \Rightarrow(3)$. We show the following statement: If $X$ is connected, metrisable, and locally compact, then it is $\sigma$-compact. Choose some metric $d$ which defines the topology. For $x \in X, r>0$, let $B_{r}(x)$ be the closed $r$-ball around $x$. Since $X$ it is locally compact, for any $x$, there is some $r>0$ such that $B_{r}(x)$ is compact. Let $r(x) \in(0, \infty]$ be defined by

$$
2 \cdot r(x)=\inf \left\{r>0 \mid B_{r}(x) \text { is compact }\right\} .
$$

If $r(x)$ is infinite for some $x$, then $X$ is $\sigma$-compact, so we may assume that $r(x)$ is finite for all $x$.

By definition, if $d(x, y) \leqslant \varepsilon$, then $2 r(x) \geqslant 2 r(y)-\varepsilon$, so that

$$
|r(x)-r(y)| \leqslant \frac{1}{2} \cdot d(x, y)
$$

In particular, $r$ is continuous. Moreover, this implies

$$
\begin{equation*}
d(x, y) \leqslant \frac{2}{3} \cdot r(x) \Rightarrow x \in B_{r(y)}(y) \tag{*}
\end{equation*}
$$

In fact,

$$
r(y) \geqslant r(x)-\frac{1}{2} \cdot d(x, y) \geqslant r(x)-\frac{1}{3} \cdot r(x)=\frac{2}{3} \cdot r(x) \geqslant d(x, y)
$$

For any subset $K \subset X$, set

$$
K^{\prime}=\bigcup_{x \in K} B_{r(x)}(x)
$$

We let $K_{0}=\left\{x_{0}\right\}$ where $x_{0} \in X$ is arbitrary, and define $K_{n+1}=K_{n}^{\prime}$ inductively. Then $Y=\bigcup_{n=0}^{\infty} K_{j}$ is open, and by ( $*$ ), it is closed. Since $X$ is connected, $X=Y$ is $\sigma$-compact once we have shown that the $K_{n}$ are compact.

We claim that for any compact $K \subset X, K^{\prime}$ is also compact. Let $x_{k} \in K^{\prime}$ be a sequence. There are $y_{k} \in K$ such that $x_{k} \in B_{r_{k}}\left(y_{k}\right)$ where $r_{k}=r\left(y_{k}\right)$. Passing to a subsequence, we may assume that $y_{k} \rightarrow y \in K$, so $r_{k} \rightarrow r=r(y)$. For any $2>q>1$, there is $n_{q}$ such that $r_{k} \leqslant q r$ for all $k \geqslant n_{q}$, so $x_{k} \in B_{q r}(y)$ for all $k \geqslant n_{q}$. Since the latter ball is compact, $\left(x_{k}\right)$ has a subsequence which converges to $x \in \bigcap_{2>q>1} B_{q r}(x)=B_{r}(x) \subset K^{\prime}$. Hence, $K^{\prime}$ is compact, and we have proved our claim.
$(3) \Rightarrow(4)$. Each compact subset of $X$ has a finite cover by second countable open subsets of $X$, so the statement follows easily.
$(4) \Rightarrow(1)$. This is the Urysohn metrisation theorem.
$(2) \Rightarrow(3)$. We show: Any connected, locally compact, paracompact space is $\sigma$-compact. There is a locally finite cover $\mathcal{U}$ of $X$ by non-void open sets of compact closure. Let $U_{0} \in \mathcal{U}$. Inductively, given $U_{0}, \ldots, U_{N}$, let $U_{N+1}, \ldots, U_{N+n}$ be the members of $\mathcal{U}$ that $\overline{U_{0}}, \ldots, \overline{U_{N}}$ intersect. Then $U=\bigcup_{j=0}^{\infty} \overline{U_{j}}=\bigcup_{j=0}^{\infty} U_{j}$ is open and $\neq \varnothing$. If $x \in \bar{U}$, then by the local finiteness of $\mathcal{U}$, for some $N$, one has $x \in \overline{\bigcup_{j=0}^{N} U_{j}}=\bigcup_{j=0}^{N} \overline{U_{j}} \subset U$. Therefore, $U$ is closed and $X=U$ is $\sigma$-compact.
$(4) \Rightarrow(2)$. We prove: Any connected, locally compact and second countable space is paracompact. Any $x \in X$ has a has an open neighbourhood with compact closure. Hence, any compact $K \subset X$ has an open neighbourhood with compact closure. Inductively, one may construct open subsets $U_{j} \subset X$, such that $\overline{U_{j}} \subset U_{j+1}$ is compact, and $X=\bigcup_{j=0}^{\infty} U_{j}$.

It will be sufficient to show that $\left(U_{n}\right)$ has a locally finite refinement. Let $K_{n}=\overline{U_{n}} \backslash U_{n-1}$ and $V_{n}=U_{n+1} \backslash \overline{U_{n-2}}$ so that $K_{n}$ is compact, $V_{n}$ is open, and $V_{n-1} \subset K_{n} \subset V_{n}$. Here, we let $U_{-1}=U_{-2}=\varnothing$.

Let $\left(W_{j}\right)$ be a countable base of the topology of $X$ (this exists, in view of the equivalence of (3) and (4)). For each $n$, there exist finitely many indices $j_{n 1}, \ldots, j_{n m_{n}}$ such that $W_{j_{n i}} \subset V_{n}$ and $K_{n} \subset \bigcup_{i=1}^{m_{n}} W_{j_{n i}}$. The collection $\left(W_{n j}\right)$ is an open locally finite refinement of $\left(U_{n}\right)$.

### 5.2. Actions and quotients.

Definition 5.5. Let $G$ be a Lie supergroup, $X$ a supermanifold. A morphism $\alpha: G \times X \rightarrow X$ is a left action of $G$ on $X$ if

$$
\alpha(1, x)=x \quad \text { and } \quad \alpha\left(g_{1} g_{2}, x\right)=\alpha\left(g_{1}, \alpha\left(g_{2}, x\right)\right)
$$

for all $x \in_{S} X, g_{1}, g_{2} \in_{S} G$. Similarly, one may define right actions. One also says that $X$ is a (left resp. right) $G$-supermanifold
An action is free and proper if the morphism

$$
(\alpha, \text { id }): G \times X \rightarrow X \times X
$$

is a closed embedding. In this case, $R=R_{\alpha}=(\alpha, \mathrm{id})(G \times X)$ is a closed submanifold, and an equivalence relation on $X$. (Similarly, $R=(\mathrm{id}, \alpha)(X \times G)$ in the case of a right action.) One defines $X / G=X / R$ in the case of a right action. In the case of a left action, one writes instead $G \backslash X$.
Proposition 5.6. Let $X$ be a free and proper $G$-supermanifold. Then the quotient $X / G$ resp. $G \backslash X$ exists, and the canonical projection $\pi$ is a surjective submersion.

Proof. In the case of a left action, note that $\pi_{2}: R \rightarrow X$ identifies under the isomorphism ( $\alpha, \mathrm{id}$ ) : $G \times X \rightarrow R$ with $p_{2}: G \times X \rightarrow X$, which is clearly a submersion. Thus, Theorem 5.3 applies.

Definition 5.7. Let $G$ be a supergroup. A closed subsupergroup is a closed submanifold $H$ which is at the same time a Lie supergroup, such that the embedding $j_{H}: H \rightarrow G$ is a morphism of supergroups.

Corollary 5.8. Let $G$ be a Lie supergroup and $H$ a closed subsupergroup. Consider the natural right action of $H$ on $G$. Then $G / H$ exists, and the canonical projection $\pi: G \rightarrow G / H$ is a surjective submersion. The quotient $G / H$ is naturally a left $G$-space.

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[^0]:    Date: SS 2011, Universität zu Köln.

