Geometric Measure Theory

Dr. rer. nat. A. Alldridge

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Contents

1 Basic Measure Theory

1.1 Measures and Measurability

Definition 1.1.1. Let X be a set. A map $\mu : \mathcal{P}(X) \to [0, \infty]$ is a *measure* on X if

$$\mu(\varnothing) = 0 \quad \text{and} \quad \mu(A) \leqslant \sum_{k=0}^{\infty} \mu(A_k) \quad \text{whenever} \quad A \subset \bigcup_{k=0}^{\infty} A_k .$$

In particular, μ is increasing and σ-subadditive.

NB. Usually, measures as defined above are referred to as *outer* measures whereas usual measures are commonly supposed to be defined on some σ-algebra of sets possibly smaller than $\mathcal{P}(X)$. Any such common measure can be extended to an outer measure.

Definition 1.1.2. If $Y \subset X$ and μ is a measure on X, then its *restriction* $\mu \llcorner Y$ is defined by $(\mu \llcorner Y)(A) = \mu(A \cap Y)$ for all $A \subset X$. It is a measure on X.

A subset $A \subset X$ is said to be μ *measurable* if $\mu = \mu \llcorner A + \mu \llcorner (X \setminus A)$, i.e.

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \quad \text{for all } B \subset X .$$

It is immediate that A is μ measurable whenever $\mu(A) = 0$, and that A is μ measurable if and only $X \setminus A$ is. Moreover, since the inequality $\leqslant$ always true, it suffices to prove $\geqslant$ to prove that A is μ measurable. In particular, it suffices to consider sets B with $\mu(B) < \infty$. Denote by $\mathfrak{M}(\mu)$ the set of μ measurable sets.

A set such that $\mu(A) = 0$ shall be called μ *negligible* or a μ *zero set*. Correspondingly, $X \setminus A$ shall be called a μ *cozero set*. Often, we shall say that some predicate P is true μ *almost everywhere* (μ a.e.) or that $P(x)$ is valid for μ *almost every* $x \in X$; by which token we shall mean that the set $\{P\} = \{x \in X | P(x)\}$ is μ cozero.

Proposition 1.1.3. Let $A_k \subset X$ be μ measurable.

(i). $\bigcup_{k=0}^{\infty} A_k$ and $\bigcap_{k=0}^{\infty} A_k$ are μ measurable.

(ii). If A_k are disjoint, then $\mu\big(\bigcup_{k=0}^{\infty} A_k\big) = \sum_{k=0}^{\infty} \mu(A_k)$.

(iii). If $A_k \subset A_{k+1}$, then $\lim_k \mu(A_k) = \mu\big(\bigcup_{k=0}^{\infty} A_k\big)$.

(iv). If $A_k \supset A_{k+1}$ and $\mu(A_0) < \infty$, then $\lim_k \mu(A_k) = \mu\big(\bigcap_{k=0}^{\infty} A_k\big)$.

Proof. Clearly, the statement in (i) on intersections follows from that on unions by taking complements in X, and similarly (iv) follows from (iii) by taking complements in A_0. First, let us prove (i) for *finite* unions. It suffices to consider the $A_0 \cup A_1$ where A_j, $j = 0, 1$, are μ measurable. For $B \subset X$,

$$\mu(B) = \mu(B \cap A_0) + \mu(B \setminus A_0)$$

$$
\begin{aligned}
&= \mu(B \cap A_0) + \mu\big((B \setminus A_0) \cap A_1\big) + \mu\big((B \setminus A_0) \setminus A_1\big) \\
&\geqslant \mu\big(B \cap (A_0 \cup A_1)\big) + \mu\big(B \setminus (A_0 \cup A_1)\big) ,
\end{aligned}
$$

so $A_0 \cup A_1$ is μ measurable.

Now to (ii). Let (A_j) be disjoint and define $B_k = \bigcup_{j=0}^{k} A_j$. Then B_k is μ measurable,

$$B_{k+1} \cap A_{k+1} = A_{k+1} \quad \text{and} \quad B_{k+1} \setminus A_{k+1} = B_k .$$

Thus,

$$\mu\Big(\bigcup\nolimits_{j=0}^{k+1} A_j\Big) = \mu(B_{k+1}) = \mu(A_{k+1}) + \mu(B_k) = \sum_{j=0}^{k+1} \mu(A_j) .$$

Therefore,

$$\sum_{j=0}^{\infty} \mu(A_j) \leqslant \mu\Big(\bigcup\nolimits_{j=0}^{\infty} A_j\Big) ,$$

and the converse inequality is trivial.

Now, (iii) follows immediately, since

$$
\begin{aligned}
\lim\nolimits_k \mu(A_k) &= \mu(A_0) + \sum_{k=0}^{\infty} \big(\mu(A_{k+1}) - \mu(A_k)\big) = \mu(A_0) + \sum_{k=0}^{\infty} \mu(A_{k+1} \setminus A_k) \\
&= \mu\Big(A_0 \cup \bigcup\nolimits_{k=0}^{\infty} (A_{k+1} \setminus A_k)\Big) = \mu\Big(\bigcup\nolimits_{k=0}^{\infty} A_k\Big) .
\end{aligned}
$$

Now to the case of infinite unions in (i). Let $B \subset X$, $\mu(B) < \infty$. Any μ measurable set is $\mu \llcorner B$ measurable. In particular, B_k , defined as above, are $\mu \llcorner B$ measurable. Hence, by (iii) and (iv),

$$
\begin{aligned}
\mu(B) = (\mu \llcorner B)(X) &= \lim\nolimits_k \big((\mu \llcorner B)(B_k) + (\mu \llcorner B)(X \setminus B_k)\big) \\
&= (\mu \llcorner B)\Big(\bigcup\nolimits_{k=0}^{\infty} B_k\Big) + (\mu \llcorner B)\Big(\bigcap\nolimits_{k=0}^{\infty} (X \setminus B_k)\Big) \\
&= \mu\Big(B \cap \bigcup\nolimits_{k=0}^{\infty} B_k\Big) + \mu\Big(\bigcup\nolimits_{k=0}^{\infty} (B \setminus B_k)\Big) ,
\end{aligned}
$$

and thus follows the assertion. □

Definition 1.1.4. A set $\mathcal{A} \subset \mathcal{P}(X)$ is a *σ-algebra* if $\varnothing \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$ whenever $A \in \mathcal{A}$, and $\bigcup_{k=0}^{\infty} A_k \in \mathcal{A}$ whenever $A_k \in \mathcal{A}$.

Thus, by proposition 1.1.3, $\mathfrak{M}(\mu)$ is a σ-algebra. By the same token, the of all μ zero (resp. μ cozero) sets is closed under countable unions and intersections, although it is usually not a σ-algebra.

The intersection of σ-algebras is a σ-algebra. Thus, for any $\mathcal{A} \subset \mathcal{P}(X)$, there exists a smallest σ-algebra containing $\mathcal{A}$.

If X is a topological space, the *Borel σ-algebra* $\mathcal{B}(X)$ is the smallest σ-algebra containing the topology of X . Sets $A \in \mathcal{B}(X)$ are called *Borel* (after E. Borel).

Definition 1.1.5. Let μ be a measure on X. μ is called

finite if $\mu(X) < \infty$,

regular if for all $A \subset X$, there exists a μ measurable $B \supset A$ so that $\mu(A) = \mu(B)$.

Now let X be a topological space. μ is said to be

Borel if all Borel sets are μ measurable

Borel regular if μ is Borel and for all $A \subset X$, there is a *Borel* $B \supset A$ so that $\mu(A) = \mu(B)$,

Radon if μ is Borel regular and finite on compacts.

We briefly study the special properties of measures satisfying these conditions.

Proposition 1.1.6. Let μ be a regular measure on the set X. If $A_k \subset A_{k+1} \subset X$ are (not necessarily measurable) subsets, then $\lim_k \mu(A_k) = \mu\big(\bigcup_{k=0}^{\infty} A_k\big)$.

Proof. Let $B_k \supset A_k$ be μ measurable, such that $\mu(A_k) = \mu(B_k)$. Let $C_k = \bigcap_{j=k}^{\infty} B_j$. Then $\mu(A_k) \leqslant \mu(C_k) \leqslant \mu(B_k)$, and thus $\mu(C_k) = \mu(A_k)$. We have $A_k \subset C_k \subset C_{k+1}$ and C_k are measurable. Thus, by proposition 1.1.3 (iii),

$$\lim_k \mu(A_k) = \mu\Big(\bigcup\nolimits_{k=0}^{\infty} C_k\Big) \geqslant \mu\Big(\bigcup\nolimits_{k=0}^{\infty} A_k\Big) \geqslant \sup_k \mu(A_k) = \lim_k \mu(A_k)\,,$$

proving the claim. □

Proposition 1.1.7. Let X be a topological space and μ a Borel regular measure on X. If $A \subset X$ is μ measurable and $\mu(A) < \infty$, then $\mu \llcorner A$ is finite and Borel regular, in particular, a Radon measure.

Proof. $\mu \llcorner A$ is a Borel measure, and $(\mu \llcorner A)(X) = \mu(A) < \infty$. Since μ is Borel regular, there exists a Borel $B \supset A$ so that $\mu(B) = \mu(A)$. Since A is μ-measurable, $B \setminus A$ is μ negligible. Thus, for all $C \subset X$,

$$(\mu \llcorner B)(C) = \mu(C \cap B \cap A) + \mu\big((C \cap B) \setminus A\big) = \mu(C \cap A) = (\mu \llcorner A)(C)\,,$$

and we may assume that A be Borel.

To see that $\mu \llcorner A$ is Borel regular, let $C \subset X$. There exists a Borel $D \supset C \cap A$ such that $\mu(D) = (\mu \llcorner A)(C)$. Let $E = D \cup (X \setminus A)$. Then E is Borel, and $E \supset C$. Moreover,

$$(\mu \llcorner A)(C) \leqslant (\mu \llcorner A)(E) = (\mu \llcorner A)(D) \leqslant \mu(D)\,,$$

and thus, $(\mu \llcorner A)(C) = (\mu \llcorner A)(D)$. □

Lemma 1.1.8. Let X be topological, and every open subset be an F_σ (e.g., X is metrisable).[1] Let μ be a Borel measure on X, and $B \subset X$ Borel.

(i). If $\mu(B) < \infty$, then for all $\varepsilon > 0$, there exists a closed $C \subset B$ so that $\mu(B \setminus C) \leqslant \varepsilon$.

(ii). If B is contained in the union of countably many μ finite open sets, then for all $\varepsilon > 0$, there exists an open $U \supset B$ so that $\mu(U \setminus B) \leqslant \varepsilon$.

Proof of (i). Note that $\nu = \mu \llcorner B$ is a finite Borel measure. Let

$$\mathcal{F} = \big\{ A \in \mathfrak{M}(\mu) \bigm| \forall \varepsilon > 0 \, \exists \, \text{closed } C \subset A \, : \, \nu(A \setminus C) \leqslant \varepsilon \big\}$$

Of course, all closed subsets of X are contained in $\mathcal{F}$. Let $(A_k) \subset \mathcal{F}$ and fix $\varepsilon > 0$. There exist closed $C_k \subset A_k$ such that $\mu(A_k \setminus C_k) \leqslant \frac{\varepsilon}{2^{k+1}}$. Then

$$\nu\Big(\bigcap\nolimits_{k=0}^\infty A_k \setminus \bigcap\nolimits_{k=0}^\infty C_k\Big) \leqslant \nu\Big(\bigcup\nolimits_{k=0}^\infty A_k \setminus C_k\Big) \leqslant \sum_{k=0}^\infty \nu(A_k \setminus C_k) \leqslant \varepsilon .$$

Since $\bigcap_{k=0}^\infty C_k$ is closed, $\bigcap_{k=0}^\infty A_k \in \mathcal{F}$. Similarly,

$$\lim\nolimits_k \nu\Big(\bigcup\nolimits_{j=0}^k A_j \setminus \bigcup\nolimits_{j=0}^k C_j\Big) = \nu\Big(\bigcup\nolimits_{k=0}^\infty A_k \setminus \bigcup\nolimits_{k=0}^\infty C_k\Big) \leqslant \nu\Big(\bigcup\nolimits_{k=0}^\infty A_k \setminus C_k\Big) \leqslant \varepsilon ,$$

so $\nu\big(\bigcup_{k=0}^\infty A_k \setminus \bigcup_{j=0}^k C_j\big) \leqslant 2\varepsilon$ for some k, and we conclude $\bigcup_{k=0}^\infty A_k \in \mathcal{F}$.

Since every open subset of X is an F_σ, $\mathcal{F}$ contains all open subsets of X. Moreover,

$$\mathcal{G} = \big\{ A \in \mathcal{F} \bigm| X \setminus A \in \mathcal{F} \big\}$$

is a σ-algebra containing the topology of X. Hence, $B \in \mathcal{B}(X) \subset \mathcal{G} \subset \mathcal{F}$, and thence our assertion.

Proof of (ii). Let $B \subset \bigcup_{k=0}^\infty U_k$ where $U_k \subset U_{k+1}$ are open, $\mu(U_k) < \infty$. Then $U_k \setminus B$ are μ finite Borel sets, so we may apply (i) to find closed subsets $C_k \subset U_k \setminus B_k$ such that $\mu\big(U_k \setminus (B \cup C_k)\big) \leqslant \frac{\varepsilon}{2^{k+1}}$. Clearly, $B \subset U := \bigcup_{k=0}^\infty (U_k \setminus C_k)$, which is open. Moreover,

$$\mu(U \setminus B) \leqslant \sum_{k=0}^\infty \mu\big(U_k \setminus (B \cup C_k)\big) \leqslant \varepsilon ,$$

which establishes the proposition. □

Theorem 1.1.9. Let X be topological, and every open subset be an F_σ (e.g., X is metrisable). Let μ be a Borel regular measure on X, and assume that $X = \bigcup_{k=0}^\infty U_k$ where $U_k \subset X$ are open and μ finite.

(i). For all $A \subset X$, $\mu(A) = \inf\big\{\mu(U) \bigm| A \subset U, \, U \text{ open}\big\}$ ('outer regularity').

[1] If (X, d) is metric, $U \subset X$ open, then $U = \bigcup_{j=1}^\infty C_j$ where $C_j = \{\operatorname{dist}(\sqcup, X \setminus U) \geqslant \frac{1}{j}\}$.

(ii). For all $A \in \mathfrak{M}(\mu)$, $\mu(A) = \sup\{\mu(C) \mid C \subset A\,,\, C \text{ closed}\}$ ('inner regularity').

(iii). If X is Hausdorff and σ-compact, we may take compacts in place of closed subsets in (ii).

Proof of (i). If $\mu(A) = \infty$, the statement is clear. Let $\mu(A) < \infty$. There exists a Borel $B \supset A$ so that $\mu(A) = \mu(B)$. Fix $\varepsilon > 0$. By lemma 1.1.8 (ii), there exists an open $U \supset B$ such that $\mu(U \setminus B) \leqslant \varepsilon$, and

$$\mu(U) = \mu(B) + \mu(U \setminus B) \leqslant \mu(B) + \varepsilon = \mu(A) + \varepsilon\,,$$

so we have the claim.

Proof of (ii). Let $A \subset X$ be μ measurable and assume first that $\mu(A) < \infty$. Then $\mu \llcorner A$ is a Radon measure by proposition 1.1.7. Because $(\mu \llcorner A)(X \setminus A) = 0$, by (i), we can obtain for given $\varepsilon > 0$ an open $U \supset (X \setminus A)$ so that $(\mu \llcorner A)(U) \leqslant \varepsilon$. The set $C = X \setminus U$ is closed and contained in A. Furthermore,

$$\mu(A) = \mu(C) + \mu(A \setminus C) = \mu(C) + (\mu \llcorner A)(U) \leqslant \mu(C) + \varepsilon\,,$$

so $\mu(A) = \sup\{\mu(C) \mid C \subset A\,,\, C \text{ closed}\}$.

Now, let $\mu(A) = \infty$. Let $K_j = U_{j+1} \setminus \bigcup_{k=0}^{j} U_k$ where $X = \bigcup_{k=0}^{\infty} U_k$, U_k being μ finite open sets. Then K_j are disjoint, Borel, μ finite, and $X = \bigcup_{j=0}^{\infty} K_j$. Thus

$$\sum_{j=0}^{\infty} \mu(A \cap K_j) = \mu\Big(\bigcup\nolimits_{j=0}^{\infty}(A \cap K_j)\Big) = \mu(A) = \infty\,.$$

Since $\mu(A \cap K_j) < \infty$, there exist closed $C_j \subset A \cap K_j$ such that $\mu(A \cap K_j) \leqslant \mu(C_j) + \frac{1}{2^j}$. Hence $\sum_{k=0}^{\infty} \mu(C_j) = \infty$, and

$$\lim\nolimits_k \mu\Big(\bigcup\nolimits_{j=0}^{k} C_j\Big) = \mu\Big(\bigcup\nolimits_{j=0}^{\infty} C_j\Big) = \sum_{j=0}^{\infty} \mu(C_j) = \infty\,.$$

This proves the equation also in the case of infinite measure.

Proof of (iii). Any compact is closed, and any closed set is the countable ascending union of compacts. □

Theorem (Carathéodory's criterion) 1.1.10. Let (X, d) be metric and μ a measure on X. If $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \subset X$ satisfy $\operatorname{dist}(A, B) > 0$, then μ is a Borel measure.

Proof. It is sufficient to show that any closed $C \subset X$ is μ measurable. Let $A \subset X$, $\mu(A) < \infty$. We claim that $\mu(A \cap C) + \mu(A \setminus C) \leqslant \mu(A)$.

Let $C_n = \{x \in X \mid \operatorname{dist}(x, C) \leqslant \frac{1}{n}\}$. Then

$$\operatorname{dist}(A \setminus C_n, A \cap C) \geqslant \operatorname{dist}(X \setminus C_n, C) \geqslant \tfrac{1}{n} > 0 .$$

Hence,

$$\mu(A \setminus C_n) + \mu(A \cap C) = \mu\big((A \setminus C_n) \cup (A \cap C)\big) \leqslant \mu(A) .$$

Thus, our claim follows as soon as $\lim_n \mu(A \setminus C_n) = \mu(A \setminus C)$. Let $D_k = (A \cap C_k) \setminus C_{k+1}$. Then $A \setminus C = A \setminus C_n \cup \bigcup_{k=n}^{\infty} D_k$, and thus

$$\mu(A \setminus C_n) \leqslant \mu(A \setminus C) \leqslant \mu(A \setminus C_n) + \sum_{k=n}^{\infty} \mu(D_k) ,$$

so it suffices to prove $\sum_{k=1}^{\infty} \mu(D_k) < \infty$. Note that $\operatorname{dist}(D_k, D_\ell) > 0$ as soon as $|k - \ell| \geqslant 2$. Therefore,

$$\begin{aligned}\sum_{k=1}^{\infty} \mu(D_k) &= \sup\nolimits_{m \geqslant 1} \Big[\sum_{k=1}^{m} \mu(D_{2k}) + \sum_{k=1}^{m} \mu(D_{2k-1})\Big] \\ &= \sup\nolimits_{m \geqslant 1} \Big[\mu\Big(\bigcup\nolimits_{k=1}^{m} D_{2k}\Big) + \mu\Big(\bigcup\nolimits_{k=1}^{m} D_{2k-1}\Big)\Big] \leqslant 2\mu(A) < \infty ,\end{aligned}$$

so the claim and hence the theorem follow. □

1.2 Support of a Measure

Definition 1.2.1. Let X be a set, μ a measure on X. If $C \subset X$, then μ is said to be *concentrated* on C if $X \setminus C$ is μ-negligible. If X is topological and C is closed, then μ is said to be *supported* on C if it is concentrated on C.

Moreover, let $\operatorname{supp} \mu$, the *support* of μ, be the set of all $x \in X$ such that $\mu(U) > 0$ for all neighbourhoods U of x. Then $\operatorname{supp} \mu$ is closed.

Proposition 1.2.2. If X is a σ-compact Hausdorff space, and μ is a Radon measure, then μ is supported on $\operatorname{supp} \mu$. (I.e., $\operatorname{supp} \mu$ is the complement of the largest open negligible subset of X.)

Proof. Let $U = X \setminus \operatorname{supp} \mu$. Let $C \subset U$ be compact. Then C is contained in a finite union of open μ negligible sets, so $\mu(C) = 0$. By theorem 1.1.9 (iii), $\mu(U) = 0$. □

Remark 1.2.3. Although it is not too easy to construct counter-examples, not every Borel regular measure is supported on its support. Compare [Fed69, ex. 2.5.15] for an example with $X = [-1, 1]^J$, J uncountable.

Definition 1.2.4. A cardinal α is called an *Ulam number* if for any measure μ on a set X, and any disjoint family $\mathcal{F}$, $\#\mathcal{F} \leqslant \alpha$, $\mu\big(\bigcup \mathcal{F}\big) < \infty$, such that $\bigcup \mathcal{G}$ is μ measurable for all $\mathcal{G} \subset \mathcal{F}$, we have

$$(\forall A \in \mathcal{F} : \mu(A) = 0) \quad \Rightarrow \quad \mu\Big(\bigcup \mathcal{F}\Big) = 0 .$$

Clearly, any finite cardinal, and also the cardinality of a countable infinity, $\aleph_0 = \#\mathbb{N}$, is an Ulam number. The statement that any cardinality of a set is an Ulam number is consistent with the ZFC axioms of set theory, cf. [Fed69, 2.1.6].

Theorem 1.2.5. Let X be metric, μ a finite Borel measure. Then $\operatorname{supp}\mu$ is separable. If X contains a dense subset Y such that $\#Y$ is an Ulam number, then μ is supported on $\operatorname{supp}\mu$.

Proof. For $A_n \subset \operatorname{supp}\mu$ be maximal with property that for any distinct $x, y \in A_n$, $d(x,y) > \frac{2}{n}$. Then $\mathcal{A}_n = \{B(x, \frac{1}{n}) \mid x \in A_n\}$ is disjoint, and

$$\textstyle\sum_{A\in\mathcal{A}_n} \mu(A) = \mu(\bigcup \mathcal{A}_n) \leqslant \mu(X) < \infty\,,$$

so A_n is countable. For all $x \in X$, there exists $y \in A_n$ such that $d(x,y) \leqslant \frac{2}{n}$. Therefore, $\bigcup_{n=1}^\infty A_n$ is dense.

W.l.o.g., assume that $\#X$ is infinite. If $\alpha = \#Y$ is an Ulam number for some dense $Y \subset X$, then the set of balls with rational radii centred on $y \in Y$ has cardinality α, and forms a base of the topology of X. Thus, there exists a family $\mathcal{U}$ of μ neglible open subsets of X, such that

$$X \setminus \operatorname{supp}\mu = \bigcup \mathcal{U} \quad \text{and} \quad \#\mathcal{U} \leqslant \alpha\,.$$

Choose a well-ordering $\preceq$ of $\mathcal{U}$.[2] For $U \in \mathcal{U}$ and $n \in \mathbb{N}\setminus 0$, let

$$C_n(U) = \{x \in X \mid V \prec U \Rightarrow x \notin V\,,\ \operatorname{dist}(x, X\setminus U) \geqslant \tfrac{1}{n}\}\,.$$

Clearly, $C_n(U)$ is closed. Moreover,

$$\bigcup \mathcal{U} = \bigcup_{n=1}^\infty \bigcup \mathcal{C}_n \quad \text{where} \quad \mathcal{C}_n = \{C_n(U) \mid U \in \mathcal{U}\}\,.$$

If $x, y \in \bigcup \mathcal{C}_n$ are such that $x \in C_n(U)$ and $y \in C_n(V)$ where $U \neq V$, we may assume $U \prec V$. Hence, $d(x,y) \geqslant \frac{1}{n}$. This implies that $\mathcal{C}_n$ is disjoint; furthermore, if $\mathcal{F} \subset \mathcal{C}_n$, then

$$x \in \bigcup \mathcal{C}_n \setminus \bigcup F \quad \Rightarrow \quad B(x, \tfrac{1}{2n}) \subset \bigcup \mathcal{C}_n \setminus \bigcup \mathcal{F}\,,$$

so $\bigcup \mathcal{F}$ is closed in $\bigcup \mathcal{C}_n$. If $x \in \bigcup \mathcal{U} \setminus \bigcup \mathcal{C}_n$, then $x \in U$ for some $U \in \mathcal{U}$, and we have $\delta = \frac{1}{2} - \operatorname{dist}(x, X\setminus U) > 0$. Thus, $B(x, \frac{\delta}{2}) \subset X \setminus C_n(U)$ and $B(x, \frac{1}{4}) \subset X \setminus C_n(V)$ for all $V \neq U$. Therefore $\bigcup \mathcal{C}_n$ is locally closed, and $\bigcup \mathcal{F}$ is μ measurable. Since $\#\mathcal{F} \leqslant \alpha$, we find $\mu(\bigcup \mathcal{C}_n) = 0$. Hence, $\mu(\bigcup \mathcal{U}) = 0$. □

[2] Recall that a well-ordering on a set P is an order such that each non-empty subset of P has a least element. Any well-ordering is total. The statement that any set can be well-ordered is equivalent to the axiom of choice.

Corollary 1.2.6. Let X be complete metric, μ a finite Borel measure. If $\#X$ is an Ulam number, then μ is concentrated on a σ-compact.

Proof. Let $(x_n) \subset \operatorname{supp}\mu$ be a dense sequence. Define $L_{k\ell} = \bigcup_{j=0}^{\ell} B\bigl(x_j, \frac{1}{k}\bigr)$. Since μ is concentrated on $\operatorname{supp}\mu$, for all $k \geqslant 1$, and any $\varepsilon > 0$, there exists $\ell_{k,\varepsilon} \in \mathbb{N}$ such that $\mu\bigl(X \setminus L_{k,\ell_{k,\varepsilon}}\bigr) \leqslant \frac{\varepsilon}{2^k}$. Let $K_\varepsilon = \bigcap_{k=1}^{\infty} L_{k,\ell_{k,\varepsilon}}$. Then $\mu(X \setminus K_\varepsilon) \leqslant \varepsilon$.

We claim that K_ε is compact. Let $(y_n) \subset K_\varepsilon$ be a sequence, and set $A_0 = \mathbb{N}$. Inductively, define infinite sets $A_{k+1} \subset A_k \subset A_0$ as follows. Since $(y_n)_{n\in A_k} \subset L_{k+1,\ell_{k+1,\varepsilon}}$, there exists some j such that $d(x_j, y_n) \leqslant \frac{1}{k+1}$ for infinitely many $n \in A_k$. Let A_{k+1} be the set of all these $n \in A_k$. Now define $\alpha(0) = 0$ and

$$\alpha(k+1) = \min\bigl\{n \in A_{k+1} \bigm| n > \alpha(k)\bigr\} .$$

Then $(y_{\alpha(k)})$ is a subsequence of (y_n), and $d(y_{\alpha(k)}, y_{\alpha(\ell)}) \leqslant \frac{1}{2k}$ for all $\ell \geqslant k \geqslant 1$. Hence, $(y_{\alpha(k)})$ is a Cauchy sequence, and therefore converges to some $y \in K_\varepsilon$.

Now, $A = \bigcup_{k=1}^{\infty} K_{1/k}$ is a σ-compact, such that $\mu(X \setminus K) = 0$. □

1.3 Measurable Functions

Definition 1.3.1. Let X be a set, Y a topological space, and μ a measure on X. A map $f : X \to Y$ is said to be μ *measurable* if $f^{-1}(U)$ is μ measurable for each open $U \subset Y$. Equivalently, $f^{-1}(B)$ is μ measurable for all Borel $B \subset Y$.

Set $\bar{\mathbb{R}} = [-\infty, \infty]$, and endow this set with the (compact, metrisable) topology generated by the set of all intervals $[-\infty, a[$ and $]a, \infty]$ for $-\infty < a < \infty$. Then a function $f : X \to \bar{\mathbb{R}}$ is μ measurable if and only if $\{f < a\}$ is μ measurable for all $a < \infty$. If in particular, X is topological and μ is a Borel measure, then all l.s.c. and u.s.c. functions are μ measurable. The characteristic function $c \cdot 1_A$ where $c \in \bar{\mathbb{R}} \setminus 0$ is μ measurable if and only if A is.

A set $A \subset X$ is said to be *σ-finite* for μ if it is measurable and the countable union of A_k, $\mu(A_k) < \infty$. Similarly, $f : X \to \bar{\mathbb{R}}$ is said to be *σ-finite* for μ if it is μ measurable and $\{f \neq 0\}$ is σ-finite.

Proposition 1.3.2.

(i). Let $f, g : X \to \mathbb{R}$ be μ measurable. Then so are $f + g$, fg, $f^{\pm}$, $|f|$, $\min(f, g)$, $\max(f, g)$. If $g \neq 0$ on X, then so is f/g.

(ii). Let $f_k : X \to \bar{\mathbb{R}}$ be μ measurable. Then so are $\inf_k f_k$, $\sup_k f_k$, $\limsup_k f_k$, $\liminf_k f_k$.

Proof of (i). Note

$$\{f + g < a\} = \bigcup_{b,c\in\mathbb{Q}\,,\,b+c<a} \{f < b\} \cap \{g < c\} ,$$

so $f+g$ is μ measurable. Because $\{f^2 < a\} = \varnothing$ if $a \leqslant 0$ and $\{f^2 < a\} = \{-b < f < b\}$ where $b^2 = a$ otherwise, f^2 is μ measurable. Since $2fg = (f+g)^2 - (f^2+g^2)$, fg is μ measurable, too.

Let $g \neq 0$ everywhere. Then

$$\{1/g < a\} = \begin{cases} \{1/a < g < 0\} & a < 0\,, \\ \{g < 0\} \cup \{1/a < g\} & a = 0\,, \end{cases}$$

where $1/0 = \infty$. Thus $1/g$ is μ measurable, and the same follows for f/g.

Now, $f^+ = 1_{f \geqslant 0} \cdot f$ and $f^- = (-f)^+$ are μ measurable. Since

$$|f| = f^+ + f^-\,,\ \max(f,g) = (f-g)^+ + g\,,\ \text{and}\ \min(f,g) = -\max(-f,-g)\,,$$

the item (i) is proven.

Proof of (ii). First, $\{\inf_k f_k < a\} = \bigcup_{k=0}^{\infty}\{f_k < a\}$. Then, $\sup_k f_k = -\inf(-f_k)$, and moreover, $\liminf_k f_k = \sup_k \inf_{j \geqslant k} f_j$, and $\limsup_k = -\liminf_k(-f_k)$. □

The following lemma will be useful later. It exhibits a positive function as the upper envelope of sums of characteristic functions which are μ measurable if f is. This is called a *pyramidal approximation*.

Lemma 1.3.3. Let $f : X \to [0,\infty]$. Then $f = \sup_k f_k$ where

$$f_k = \frac{1}{2^k} \cdot \sum_{j=1}^{k \cdot 2^k} 1_{\{f > j/2^k\}} \quad \text{for all}\ \ k \in \mathbb{N}.$$

Proof. By definition, $f_k < f$. On the other hand, for $k \geqslant f(x)$,

$$2^k \cdot \big(f(x) - f_k(x)\big) = 2^k \cdot f(x) - \lfloor 2^k \cdot f(x) \rfloor + 1 \leqslant 2\,,$$

so $0 \leqslant f(x) - f_k(x) \leqslant \frac{1}{2^{k-1}}$. □

1.4 Lusin's and Egorov's Theorems

Theorem (Lusin) 1.4.1. Let X, Y be metric with X σ-compact and Y separable, and let μ be a Borel regular measure on X. Let $A \subset X$ be μ measurable and $\mu(A) < \infty$. For any $\varepsilon > 0$, there exists a compact $K \subset A$ such that $\mu(A \setminus K) \leqslant \varepsilon$ and $f \mid K$ is continuous.

Proof. For any $k \geqslant 1$, let $(B_{k\ell})_{\ell \in \mathbb{N}} \subset \mathcal{B}(Y)$ be a Borel partition of Y so that $\operatorname{diam} B_{k\ell} \leqslant \frac{1}{k}$. Let $A_{k\ell} = A \cap f^{-1}(B_{k\ell})$. Then $(A_{k\ell})_{\ell \in \mathbb{N}} \subset \mathfrak{M}(\mu)$ is a μ measurable partition of A.

Consider $\nu = \mu \llcorner A$, a Radon measure by proposition 1.1.7. Thus X is the countable union of ν-finite open sets. Theorem 1.1.9 gives compact subsets $K_{k\ell} \subset A_{k\ell}$ satisfying

$\nu(A_{k\ell} \setminus K_{k\ell}) \leqslant \frac{\varepsilon}{2^{k+\ell+1}}$. Thus

$$\lim_n \mu\Big(A \setminus \bigcup_{\ell=0}^{n} K_{k\ell}\Big) = \nu\Big(A \setminus \bigcup_{\ell=0}^{\infty} K_{k\ell}\Big) \leqslant \nu\Big(\bigcup_{\ell=0}^{\infty} A_{k\ell} \setminus K_{k\ell}\Big) \leqslant \sum_{\ell=0}^{\infty} \nu(A_{k\ell} \setminus K_{k\ell}) \leqslant \frac{\varepsilon}{2^k} .$$

Hence, for some $n_k \in \mathbb{N}$ and $C_k = \bigcup_{\ell=0}^{n_k} K_{k\ell}$, we have $\mu(A \setminus C_k) \leqslant \frac{\varepsilon}{2^k}$.

Define $g_k : C_k \to Y$ by $g_k = b_{k\ell}$ on $K_{k\ell}$, $\ell = 0 \dots, n_k$, where $b_{k\ell} \in B_{k\ell}$ are chosen arbitrarily. Since $\operatorname{diam} B_{k\ell} \leqslant \frac{1}{k}$, we find $d(f, g_k) \leqslant \frac{1}{k}$ on C_k . Let $K = \bigcap_{k=1}^{\infty} C_k$. Then

$$\mu(A \setminus K) \leqslant \sum_{k=1}^{\infty} \mu(A \setminus C_k) \leqslant \varepsilon .$$

Moreover, $\lim_k g_k = f$ uniformly on K and the g_k are locally constant, so $f \mid K$ is continuous. □

Corollary 1.4.2. Let X be metric and σ-compact, μ a Borel regular measure on X , E locally convex and metrisable, and $f : X \to E$ be μ measurable. If $A \in \mathfrak{M}(\mu)$, $\mu(A) < \infty$, and $\varepsilon > 0$, then there is a continuous function $g : X \to E$ such that $(\mu \llcorner A)\{f \neq g\} \leqslant \varepsilon$.

Proof. By theorem 1.4.1, there exists a compact $K \subset A$ such that $\mu(A \setminus K) \leqslant \varepsilon$ and $f \mid K$ continuous. By the Dugundji-Tietze theorem [Dug51, th. 4.1], there is a continuous extension $g : X \to E$ of $f \mid K$. Since $\{f \neq g\} \subset X \setminus K$, the assertion follows. □

Remark 1.4.3. An equivalent formulation of the conditions on E in the above corollary is the following: E is locally convex, Hausdorff and first countable (i.e. 0 has a countable neighbourhood basis). The reader may be more familiar with the classical Tietze extension theorem, where $E = \mathbb{R}$. Of course, the classical Tietze theorem gives the above corollary for $E = \mathbb{R}^n$ with its Hausdorff vector space topology.

Theorem (Egorov) 1.4.4. Let X be a set, Y be metric, μ a measure on X , and $f, f_k : X \to Y$ be μ measurable. If $A \in \mathfrak{M}(\mu)$, $\mu(A) < \infty$, and $f = \lim_k f_k$ pointwise μ a.e. on A , then for all $\varepsilon > 0$ there exists μ measurable $B \subset A$ such that $\mu(A \setminus B) \leqslant \varepsilon$ and $f = \lim_k f_k$ uniformly on B .

Proof. Let $A_{k\ell} = \bigcup_{j=k}^{\infty} \{d(f_k, f) \geqslant \frac{1}{\ell}\}$. Then $A_{k+1,\ell} \subset A_{k\ell}$, and $A \cap \bigcap_{k=0}^{\infty} A_{k\ell}$ is μ negligible for all $\ell \geqslant 1$. Thus, $\lim_k \mu(A \cap A_{k\ell}) = 0$.

Fix $\varepsilon > 0$. For all ℓ , there exists $n_\ell \geqslant 1$ such that $\mu(A \cap A_{n_\ell,\ell}) \leqslant \frac{\varepsilon}{2^\ell}$. Then the set $N = \bigcup_{\ell=1}^{\infty} (A \cap A_{n_\ell,\ell})$ is μ measurable and $\mu(N) \leqslant \varepsilon$. If $x \in B = A \setminus N$, then for all $k \geqslant n_\ell$, we have $d\big(f_k(x), f(x)\big) \leqslant \frac{1}{\ell}$, so $f = \lim_k f_k$ uniformly on B . □

Corollary 1.4.5. Let X be a set, Y be metric, μ a measure on X , and $f, f_k : X \to Y$ be μ measurable. Let $A \in \mathfrak{M}(\mu)$ be σ-finite and $f = \lim_j f_j$ pointwise μ a.e. on A . Then there exist $\mathfrak{M}(\mu) \ni B_k \subset A$ such that $\mu\big(A \setminus \bigcup_{k=0}^{\infty} B_k\big) = 0$ and $f = \lim_j f_j$ uniformly on B_k .

Proof. Let $A = \bigcup_{k=0}^{\infty} A_k$ where $A_k \in \mathfrak{M}(\mu)$, $\mu(A_k) < \infty$. We may assume $A_k \subset A_{k+1}$. There exist μ measurable $B_k \subset A_k$ such that $\mu(A_k \setminus B_k) \leqslant \frac{1}{2^{k+1}}$ and $f = \lim_k f_k$ uniformly on B_k. We may assume $B_k \subset B_{k+1}$. Thus for all $\ell \in \mathbb{N}$,

$$\mu\Big(A \setminus \bigcup_{k=0}^{\infty} B_k\Big) = \mu\Big(\bigcup_{k=\ell}^{\infty} A_k \setminus \bigcup_{k=\ell}^{\infty} B_k\Big) \leqslant \mu\Big(\bigcup_{k=\ell}^{\infty} A_k \setminus B_k\Big) \leqslant \frac{1}{2^\ell}\,.$$

We conclude that $\mu\big(A \setminus \bigcup_{k=0}^{\infty} B_k\big) = 0$. □

1.5 Integrals and Limit Theorems

Definition 1.5.1. Let X be a set. A function $f : X \to \bar{\mathbb{R}}$ is called *simple* if its image is countable. (This means $f = \sum_{k=0}^{\infty} a_k 1_{A_k}$ where the A_k are mutually disjoint.) Note that summation on $\bar{\mathbb{R}}$ is well-defined, associative and commutative for all pairs $x, y \in \bar{\mathbb{R}}$ such that $\{x, y\} \neq \{\pm\infty\}$. Thus for a measure μ on X we may define

$$\int f\,d\mu = \sum_{0 \leqslant y \leqslant \infty} y \cdot \mu\big(g^{-1}(y)\big) \quad \text{for all simple, } \mu \text{ measurable } f : X \to [0, \infty]$$

and

$$\int f\,d\mu = \int f^+\,d\mu - \int f^-\,\mu = \sum_{-\infty \leqslant y \leqslant \infty} y \cdot \mu\big(f^{-1}(y)\big)$$

for all simple, μ measurable $f : X \to \bar{\mathbb{R}}$ for which $\int f^+\,d\mu < \infty$ or $\int f^-\,d\mu < \infty$.[3] In the latter case, f is called a μ *integrable* simple function. (Not to be confused with the notion of μ-summable function, to be defined below.)

Definition 1.5.2. Let $f : X \to \bar{\mathbb{R}}$ be arbitrary. Define the *upper* and *lower integral* of f by

$$\int^* f\,d\mu = \inf_{f \leqslant g\ \mu\text{ a.e.}, g\ \mu\text{-integrable simple}} \int g\,d\mu \quad \text{and} \quad \int_* f\,d\mu = -\int^* (-f)\,d\mu\,.$$

If f is μ measurable and $\int^* f\,d\mu = \int_* f\,d\mu$, then f is called μ *integrable*. In this case, the *integral* of f is, per definition,

$$\int f\,d\mu := \int^* f\,d\mu = \int_* f\,d\mu\,.$$

If $A \subset X$ is μ measurable, define $\int_A f\,d\mu = \int f\,d(\mu \llcorner A)$ whenever f is $\mu \llcorner A$-integrable.

If f is μ integrable and $\int |f|\,d\mu < \infty$, we say that f is μ *summable* (see below). If X is topological, then f is said to be *locally μ summable* if f μ measurable and $\mu \llcorner U$ summable for U in a neighbourhood basis of X. E.g., for X metric, on all balls, or for X locally compact, for all compacts. A locally μ summable function is always $\mu \llcorner K$-summable for each compact $K \subset X$.

The terminology integrable/summable is Federer's. An alternative nomenclature which often appears in the literature is quasi-integrable/integrable.

[3] For $A \subset \bar{\mathbb{R}}$, $\sum_{a \in A} a := \Sigma^+ - \Sigma^-$ if either of $\Sigma^\pm$ is finite. Here, $\Sigma^\pm = \sup_{F \subset A, \#F < \infty} \sum_{a \in F} a^\pm$.

Remark 1.5.3. Note that any μ measurable $f \geqslant 0$ (μ a.e.) is μ integrable.

Indeed, if $\mu\{f = \infty\} > 0$, then

$$\int^* f\, d\mu \geqslant \int_* f\, d\mu \geqslant \int \infty \cdot 1_{\{f=\infty\}}\, d\mu = \infty \cdot \mu\{f = \infty\} = \infty\,.$$

Otherwise, let $1 < t < \infty$, and define

$$g = \sum_{k=0}^{\infty} t^n \cdot 1_{\{t^n \leqslant f < t^{n+1}\}}\,.$$

Then $g \leqslant f \leqslant t \cdot g$, so $\int^* f\, d\mu \leqslant t \cdot \int g\, d\mu \leqslant t \cdot \int_* f\, d\mu$. Since t was arbitrary, the assertion follows.

Definition 1.5.4. Let X be topological. A set function $\nu : \mathcal{P}(X) \to \bar{\mathbb{R}}$ is a *signed measure* if there is a Radon measure μ and a locally μ summable function $f : X \to \bar{\mathbb{R}}$ so that

$$\nu(K) = \int_K f\, d\mu \quad \text{for all compacts } K \subset X\,.$$

In this case, we write $\nu = \mu \llcorner f$. Note that $\mu \llcorner A = \mu \llcorner 1_A$ if X is σ-compact and 1_A is locally μ-summable.

Theorem (Fatou's lemma) 1.5.5. Let X be a set, μ a measure on X, and $f_k : X \to [0, \infty]$ be μ-measurable. Then

$$\int \liminf_k f_k\, d\mu \leqslant \liminf_k \int f_k\, d\mu\,.$$

Proof. Let $g \leqslant \liminf_k f_k$, $g = \sum_{k=0}^{\infty} a_k 1_{A_k}$ with $A_k \in \mathfrak{M}(\mu)$, be a μ-integrable simple function. We may assume A_k to be disjoint, and thus $a_k > 0$ for all $k \in \mathbb{N}$. Let $0 < t < 1$. Then

$$A_k = \bigcup\nolimits_{\ell=0}^{\infty} B_{k\ell} \quad \text{where} \quad B_{k\ell} = A_k \cap \bigcap\nolimits_{j=\ell}^{\infty} \{f_j > t a_k\}\,.$$

Indeed, if $x \in A_k$, then $a_k = g(x) \leqslant \sup_\ell \inf_{j \geqslant \ell} f_j(x)$, so for all $\varepsilon > 0$ there exists $\ell \in \mathbb{N}$ such that $a_k - \varepsilon < \sup_{j \geqslant \ell} f_j(x)$. If we choose $0 < \varepsilon \leqslant (1-t) \cdot a_k$, then $f_j(x) > t a_k$ for all $j \geqslant \ell$. Since $A_k \supset B_{k,\ell+1} \supset B_{k,\ell}$, we have

$$\int f_\ell\, d\mu \geqslant \sum_{k=0}^{\infty} \int_{A_k} f_\ell\, d\mu \geqslant \sum_{k=0}^{\infty} \int_{B_{k\ell}} f_\ell\, d\mu \geqslant \sum_{k=0}^{\infty} t a_k \mu(B_{k\ell})\,.$$

Therefore,

$$\liminf_\ell \int f_\ell\, d\mu \geqslant \sup_\ell \sum_{k=0}^{\infty} t a_k \mu(B_{k\ell}) = t \cdot \sum_{k=0}^{\infty} a_k \mu(A_k) = t \cdot \int g\, d\mu\,.$$

Since t and g were arbitrary, the assertion follows. ☐

Corollary 1.5.6. Let $0 \leqslant f_k \leqslant f_{k+1}$ μ a.e. Then $\int \sup_k f_k\, d\mu = \sup_k \int f_k\, d\mu$.

Proof. Clearly, $\sup_k \int f_k \, d\mu \leqslant \int \sup_k f_k \, d\mu$. The converse follows from Fatou's lemma, because $\liminf_k = \sup_k$ for increasing sequences. □

Theorem (Lebesgue's dominated convergence) 1.5.7. Let $f, f_k : X \to \bar{\mathbb{R}}$ be μ measurable. Suppose $|f_k| \leqslant g$ where g is μ summable (i.e. μ measurable and $\int g \, d\mu < \infty$), and $f = \lim_k f_k$ μ a.e. Then f is μ summable and

$$\lim_k \int |f - f_k| \, d\mu = 0 \, .$$

Proof. Clearly, $|f - f_k| = \lim_\ell |f_\ell - f_k| \leqslant 2g$. By theorem 1.5.5,

$$\begin{aligned} \int 2g \, d\mu &= \int \liminf_k \big(2g - |f - f_k|\big) \, d\mu \\ &\leqslant \liminf_k \int \big(2g - |f - f_k|\big) \, d\mu = \int 2g \, d\mu - \limsup_k \int |f - f_k| \, d\mu \, , \end{aligned}$$

so

$$0 \leqslant \liminf_k \int |f - f_k| \, d\mu \leqslant \limsup_k \int |f - f_k| \, d\mu \leqslant 0 \, .$$

Since $|f| \leqslant g$, f is μ summable. □

Theorem (Pratt) 1.5.8. Let $g, g_k : X \to [0, \infty]$ be μ summable, $f, f_k : X \to \bar{\mathbb{R}}$ be μ measurable,

$$|f_k| \leqslant g_k \, , \; f = \lim_k f_k \text{ and } g = \lim_k g_k \quad \mu \text{ a.e.}$$

If $\lim_k \int g_k \, d\mu = \int g \, d\mu$, then f is μ summable, and

$$\lim_k \int |f - f_k| \, d\mu = 0 \, .$$

Proof. The proof is similar as for Lebesgue's theorem: Indeed,

$$\begin{aligned} \int 2g \, d\mu &= \int \liminf_k \big(2g_k - |f - f_k|\big) \, d\mu \\ &\leqslant \liminf_k \int \big(2g_k - |f - f_k|\big) \, d\mu \leqslant \int 2g \, d\mu - \limsup_k \int |f - f_k| \, d\mu \end{aligned}$$

by theorem 1.5.5, and again the claim follows. □

Theorem (Riesz-Fischer) 1.5.9. Let $f, f_k : X \to \bar{\mathbb{R}}$ be μ summable, $\lim_k \int |f - f_k| \, d\mu = 0$. Then there exists a subsequence α such that $f = \lim_k f_{\alpha(k)}$ μ a.e.

Proof. Let α be a subsequence such that $\int |f - f_{\alpha(k)}| \, d\mu \leqslant \frac{1}{2^{k+1}}$. Let $\varepsilon > 0$. Then $\limsup_k |f(x) - f_{\alpha(k)}| > \varepsilon$ implies that $\sup_{\ell \geqslant k} |f(x) - f_{\alpha(\ell)}(x)| > \varepsilon$ for all k, so for all k, there exists $\ell \geqslant k$ such that $|f(x) - f_{\alpha(\ell)}(x)| > \varepsilon$. Hence, for all $k \in \mathbb{N}$,

$$\mu\big[\limsup_j |f - f_{\alpha(j)}| > \varepsilon\big] \leqslant \sum_{\ell=k}^{\infty} \mu\big[|f - f_{\alpha(\ell)}| > \varepsilon\big] \leqslant \frac{1}{\varepsilon} \cdot \sum_{\ell=k}^{\infty} \int |f - f_{\alpha(\ell)}| \, d\mu \leqslant \frac{1}{2^k \cdot \varepsilon} \, .$$

Thus, $\mu\{\limsup_k |f - f_{\alpha(k)}| > 0\} = \lim_{\varepsilon\to 0+} \mu\{\limsup_k |f - f_{\alpha(k)}| > \varepsilon\} = 0$. ——— □

1.6 Product Measures

1.6.1. Let μ and ν be measures on the sets X and Y, respectively. Define a set function $\alpha \otimes \beta : \mathcal{P}(X \times Y) \to [0,\infty]$ by

$$(\mu \otimes \nu)(C) = \inf\Big\{ \mu(A_j)\nu(B_j) \;\Big|\; C \subset \textstyle\bigcup_{j=0}^{\infty} A_j \times B_j\,,\ A_j \in \mathfrak{M}(\mu)\,,\ B_j \in \mathfrak{M}(\nu) \Big\}\,.$$

Here, we let $\infty \cdot 0 = 0 \cdot \infty = 0$. It is easy to see that $\mu \otimes \nu$ is a measure on $X \times Y$, and

$$(\mu \otimes \nu)(A \times B) \leqslant \mu(A) \cdot \nu(B) \quad \text{for all } A \subset X\,,\ B \subset Y\,.$$

For $C \subset X \times Y$ and $(x,y) \in X \times Y$, let $C_x = \mathrm{pr}_2((x \times Y) \cap C)$ and $C_y = \mathrm{pr}_1((X \times y) \cap C)$.

Theorem (Fubini) 1.6.2.

(i). $\mu \otimes \nu$ is a regular measure on $X \times Y$.

(ii). The set $A \times B$ is $\mu \otimes \nu$ measurable whenever A and B are μ and ν measurable, respectively, and in this case, $(\mu \otimes \nu)(A \times B) = \mu(A) \cdot \mu(B)$.

(iii). If $C \subset X \times Y$ is σ-finite for $\mu \otimes \nu$, then C_x is ν measurable for μ a.e. x, C_y is μ measurable for ν a.e. y, $x \mapsto \nu(C_x)$ is μ integrable, $y \mapsto \mu(C_y)$ is ν integrable, and

$$(\mu \otimes \nu)(C) = \int \mu(C_y)\, d\nu(y) = \int \nu(C_x)\, d\mu(x)\,.$$

(iv). If $f : X \times Y \to \bar{\mathbb{R}}$ is $\mu \otimes \nu$-measurable and σ-finite for $\mu \otimes \nu$, then $\int f(\sqcup, y)\, d\nu(y)$ is μ integrable, $\int f(x, \sqcup)\, d\mu(x)$ is ν integrable, and

$$\int f\, d(\mu \otimes \nu) = \iint f(x,y)\, d\mu(x)\, d\nu(y) = \iint f(x,y)\, d\nu(y)\, d\mu(x)\,.$$

Proof. Let $\mathcal{F} \subset \mathcal{P}(X \times Y)$ consist of those C for which $x \mapsto 1_C(x,y)$ is μ integrable for all $y \in Y$, and $y \mapsto \int 1_C(x,y)\, d\mu(x)$ is ν integrable. For $C \in \mathcal{F}$, write

$$\varrho(C) = \iint 1_C(x,y)\, d\mu(x)\, d\nu(y) \in \bar{\mathbb{R}}\,.$$

From corollary 1.5.6, we find that for any disjoint family $(C_j) \subset \mathcal{F}$, $\bigcap_{j=0}^{\infty} C_j \in \mathcal{F}$, and $\varrho(\bigcap_{j=0}^{\infty} C_j) = \sum_{j=0}^{\infty} \varrho(C_j)$. From Lebesgue's theorem 1.5.7, for any increasing $(C_j) \subset \mathcal{F}$ such that $\varrho(C_0) < \infty$, $\bigcup_{j=0}^{\infty} C_j \in \mathcal{F}$, and $\lim_j \varrho(C_j) = \varrho(\bigcup_{j=0}^{\infty} C_j)$.

Further, define

$$\mathcal{P}_0 = \big\{ A \times B \;\big|\; (A,B) \in \mathfrak{M}(\mu) \times \mathfrak{M}(\nu) \big\}\,,\ \mathcal{P}_1 = \big\{ \textstyle\bigcup \mathcal{G} \;\big|\; \mathcal{G} \subset \mathcal{P}_0 \text{ countable} \big\}\,,$$

$$\mathcal{P}_2 = \left\{ \bigcap \mathcal{G} \,\middle|\, \varnothing \neq \mathcal{G} \subset \mathcal{P}_1 \text{ countable} \right\} .$$

For $A \times B \in \mathcal{P}_0$, $1_{A\times B}(x,y) = 1_A(x) \cdot 1_B(y)$, so $A \times B \in \mathcal{F}$ with $\varrho(A \times B) = \mu(A) \cdot \nu(B)$. In particular, $\mathcal{P}_0 \subset \mathcal{F}$. If $C \times D \in \mathcal{P}_0$, then

$$(A \times B) \setminus (C \times D) = (A \setminus C) \times B \cup (A \cap C) \times (B \setminus D) ,$$

so any element of $\mathcal{P}_1$ is the union of a *disjoint* countable family in $\mathcal{P}_0$. Thus, $\mathcal{P}_1 \subset \mathcal{F}$. Since $\mathcal{P}_1$ is closed under finite intersections, any member of $\mathcal{P}_2$ is the intersection of a decreasing sequence from $\mathcal{P}_1$.

We claim that for all $C \subset X \times Y$, there exists $C \subset D \in \mathcal{P}_2$, so that

$$(\mu \otimes \nu)(C) = \inf\{\varrho(D) | C \subset D \in \mathcal{P}_1\} \text{ , and } (\mu \otimes \nu)(C) = (\mu \otimes \nu)(D) = \varrho(D) .$$

To that end, let $A_j \times B_j \in \mathcal{P}_0$, $C \subset D = \bigcup_{j=0}^{\infty} A_j \times B_j$. Then $\varrho(D) \leqslant \sum_{j=0}^{\infty} \mu(A_j)\nu(B_j)$ with equality if the $A_j \times B_j$ are disjoint, and

$$(\mu \otimes \nu)(C) \leqslant \sum_{j=0}^{\infty} (\mu \otimes \nu)(A_j \times B_j) \leqslant \sum_{j=0}^{\infty} \mu(A_j)\nu(B_j) .$$

This establishes the first part of the claim, seeing that $(\mu \otimes \nu)(C)$ is defined as the infimum of the latter sums.

To prove the second part of the claim, we first establish (ii). If $A \times B \in \mathcal{P}_0$, then

$$(\mu \otimes \nu)(A \times B) \leqslant \mu(A)\nu(B) = \varrho(A \times B) \leqslant \varrho(C) \quad \text{for all } A \times B \subset C \in \mathcal{P}_1 .$$

By the first part of the claim, $(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B)$. For the $\mu \otimes \nu$ measurability of $E = A \times B$, let $C \subset X \times Y$. If $C \subset D \in \mathcal{P}_1$, we have $D \setminus E, D \cap E \in \mathcal{P}_1$, and

$$(\mu \otimes \nu)(C \setminus E) + (\mu \otimes \nu)(C \cap E) \leqslant \varrho(D \setminus E) + \varrho(D \cap E) = \varrho(D) ,$$

so by the first part of the claim, the measurability of $A \times B$ follows. Since $\mu \otimes \nu = \varrho$ on $\mathcal{P}_0$, this implies $\mu \otimes \nu = \varrho$ on $\mathcal{P}_1$, by proposition 1.1.3 (ii).

W.l.o.g., $(\mu \otimes \nu)(C) < \infty$, since otherwise $(\mu \otimes \nu)(C) = \infty = \varrho(X \times Y)$, and we observe $X \times Y \in \mathcal{P}_1 \subset \mathcal{P}_2$. According the first part of the claim, to $k \geqslant 1$ we can associate $C \subset C_j \in \mathcal{P}_1$ such that $\varrho(C_j) \leqslant (\mu \otimes \nu)(C) + \frac{1}{k}$. Then $D = \bigcap_{k=1}^{\infty} C_k \in \mathcal{P}_2$, and $(\mu \otimes \nu)(C) = \varrho(D)$. Since any member of $\mathcal{P}_2$ is measurable, we find $\varrho(D) = (\mu \otimes \nu)(D)$, by proposition 1.1.3 (iv). This completes the proof of the claim, and also of (i), since any $C \subset X \times Y$ is contained in an element of $\mathcal{P}_2$ of equal measure, and by (ii), these are measurable.

As to (iii), if $(\mu \otimes \nu)(C) = 0$, then there exists $C \subset D \in \mathcal{P}_2$ such that $\varrho(D) = 0$, so $\varrho(C) = 0$. If $(\mu \otimes \nu)(C) < \infty$ and C is $\mu \otimes \nu$ measurable, then there is $C \subset D \in \mathcal{P}_2$

so that $(\mu \otimes \nu)(C) = (\mu \otimes \nu)(D) = \varrho(D)$. Thus, $(\mu \otimes \nu)(D \setminus C) = 0$, in particular, $\varrho(D \setminus C) = 0$. This implies that

$$\mu(C_y) = \mu(D_y) \quad \text{for } \nu \text{ a.e.} \quad y \in Y ,$$

in particular, this quantity is ν integrable as function of y. Furthermore,

$$\varrho(C) = \varrho(D) = (\mu \otimes \nu)(C) = \int 1_C \, d(\mu \otimes \nu) .$$

The remainder of (iii) for the case of finite measure follows by symmetry. The case of σ-finite measure is a matter of applying corollary 1.5.6.

For the case of simple functions, (iv) follows immediately from (iii). If $f \geqslant 0$, the assertion follows from lemma 1.3.3 and corollary 1.5.6. The general case follows by considering $f = f^+ - f^-$. ☐

1.6.3. The construction of the product measure allows us to define the (outer) *Lebesgue measure* in arbitrary finite dimensions. The one-dimensional Lebesgue measure $\mathcal{L}^1$ is given by

$$\mathcal{L}^1(A) = \inf\left\{ \sum_{j=0}^{\infty} \operatorname{diam} A_j \;\middle|\; A \subset \bigcup_{j=0}^{\infty} A_j \,,\; A_j \subset \mathbb{R} \right\} \quad \text{for all } \; A \subset \mathbb{R} ,$$

where the diameter is taken with respect to the Euclidean distance and $\operatorname{diam} \varnothing = 0$. The n-dimensional Lebesgue measure is defined inductively by $\mathcal{L}^n = \mathcal{L}^{n-1} \otimes \mathcal{L}^1$. Then $\mathcal{L}^k \otimes \mathcal{L}^\ell = \mathcal{L}^{k+\ell}$.

Note $\mathcal{L}^1([a,b]) = b - a$ for $-\infty \leqslant a \leqslant b \leqslant \infty$. Indeed, suffices to consider a, b finite; then $\operatorname{diam}[a,b] = b - a$. Let $a_j = \inf A_j$, $b_j = \sup_j A_j$, where $[a,b] \subset \bigcup_{j=0}^{\infty} A_j$, and we may assume $A_j \cap [a,b] \neq \varnothing$ and $-\infty < a_j \leqslant b_j < \infty$. Then $a_i \leqslant a$ for some i and $b_j \geqslant b$ for some j. Moreover, $b_i \geqslant a$ and $a_j \leqslant b$, so there exist $j_0 = i, \dots, j_m = j$, such that $b_{j_k} \geqslant a_{j_{k+1}}$ for all k. Then

$$\sum_{k=0}^{\infty} \operatorname{diam}(A_k) = \sum_{k=0}^{\infty} b_k - a_k \geqslant \sum_{k=0}^{m} b_{j_k} - a_{j_k} \geqslant b_j - a_i \geqslant b - a .$$

Fubini's theorem 1.6.2 implies $\mathcal{L}^n\left(\prod_{j=1}^n [a_j, b_j]\right) = \prod_{j=1}^n b_j - a_j$ for all $-\infty \leqslant a_j \leqslant b_j \leqslant \infty$.

By Carathéodory's criterion, it follows easily that $\mathcal{L}^n$ is a Borel measure. By its σ-finiteness, the Borel regularity follows form the Borel regularity on cubes.

1.7 Covering Theorems of Vitali Type

Since the subject matter of the following two subsections is rather technical in nature, a few words on its significance are in order. In the problems we shall be studying in the sequel, passages from local to global data, and vice versa, will often be of vital importance. The technical device commonly used for such arguments will be the following: To estimate $\mu(A)$ for some set A, cover A by smaller sets in some collection $\mathcal{F}$. Then choose countable subfamilies $\mathcal{G} \subset \mathcal{F}$ almost covering A, where the members of $\mathcal{G}$ have some prescribed diameter. Conversely, if the members of $\mathcal{G}$ belong to some small neighbourhood of A and are disjoint or posess a finite bound on the number of their mutual intersections, their measure may provide a lower bound for $\mu(A)$.

The covering theorems we shall prove concern the existence of such subfamilies. Roughly, the theorems we shall prove fall into two categories: Those of Vitali type apply to quite general families $\mathcal{F}$, but provide subfamilies $\mathcal{G}$ such that certain *enlargements* of elements of $\mathcal{G}$ (almost) cover A. The Besicovich type covering theorems, on the other hand, apply to less general families $\mathcal{F}$ but provide covers by the original subfamilies $\mathcal{G}$.

1.7.1. In what follows, let (X, d) be a metric space, and μ a Borel measure.

Definition 1.7.2. Let $\mathcal{F} \subset \mathcal{P}(X)$. For each $Y \subset X$, define $\mathcal{F} \mid Y = \{F \in \mathcal{F} | F \subset Y\}$.

We say that $\mathcal{F}$ is *fine at* $x \in X$ if $\inf_{x \in F \in \mathcal{F}} \operatorname{diam}(F) = 0$, and *fine* if it is fine at every point. If $A \subset X$, we say that $\mathcal{F}$ *μ almost covers* A if $A \setminus \bigcup \mathcal{F}$ is μ negligible. The family $\mathcal{F}$ is *μ adequate for* A if for every open $V \subset X$, there exists a countable disjoint $\mathcal{G} \subset \mathcal{F}$ such that $\bigcup \mathcal{G} \subset V$ and $\mathcal{G}$ μ almost covers $V \cap A$.

Theorem 1.7.3. Let the family $\mathcal{F}$ consist of closed sets, $A = \bigcup_{j=0}^{\infty} A_j$ be σ-finite, and $\sigma : \{A_j | j \in \mathbb{N}\} \to]0, 1[$. Then $\mathcal{F}$ is μ adequate for A if the following holds: For every open $V \subset X$, there exists a countable disjoint $\mathcal{G} \subset \mathcal{F}$ such that $\bigcup \mathcal{G} \subset V$ and

$$\mu\big((V \cap A_j) \setminus \bigcup \mathcal{G}\big) \leqslant \sigma(A_j) \cdot \mu(V \cap A_j) \quad \text{for all } j \in \mathbb{N} .$$

Proof. Since A is σ-finite, we may assume that the $\mu(A_j) < \infty$ are bounded. Let (B_j) be an enumeration of $\{A_j | j \in \mathbb{N}\}$ such that for all j, $B_i = A_j$ for infinitely many i.

Fix an open $V \subset X$, We claim that for all $j \in \mathbb{N}$, there exist open $V_j \subset X$ and finite disjoint $\mathcal{G}_j \subset \mathcal{F}$ such that $\bigcup \mathcal{G}_j \subset V_j$ and

$$\mu\big((V_j \cap B_j) \setminus \bigcup \mathcal{G}_j\big) \leqslant \sigma(B_j)^{1/2} \cdot \mu(V_j \cap B_j) \quad \text{for all } j \in \mathbb{N} .$$

To that end, we set $V_0 = V$ and $\mathcal{G}_0 = \varnothing$. Moreover, set $V_j = V_{j-1} \setminus \bigcup \mathcal{G}_{j-1}$ in the j^{th} step. There exists a countable disjoint $\mathcal{H} \subset \mathcal{F}$ such that $\bigcup \mathcal{H} \subset V_j$, and

$$\mu\big((V_j \cap B_j) \setminus \bigcup \mathcal{H}\big) \leqslant \sigma(B_j) \cdot \mu(V_j \cap B_j) .$$

Since $\sigma(B_j)^{1/2} > \sigma(B_j)$, we may choose $\mathcal{G}_j \subset \mathcal{H}$ to be an appropriate finite subfamily. (Note that the members of $\mathcal{H}$ are μ measurable since they are closed and μ is Borel.) This proves the claim.

Now, let $\mathcal{G} = \bigcup_{j=0}^{\infty} \mathcal{G}_j \subset \mathcal{F}$. Then $\mathcal{G}$ is countable and disjoint,

$$\bigcup \mathcal{G} \subset \bigcup_{j=0}^{\infty} V_j \subset V \quad \text{and} \quad V \setminus \bigcup \mathcal{G} = \bigcap_{j=0}^{\infty} V_j \setminus \bigcup \mathcal{G}_j = \bigcap_{j=1}^{\infty} V_j \,.$$

If k is fixed, then $A_k = B_j$ for infinitely many j, hence

$$\mu(V_{j+1} \cap A_k) = \mu\big((A_k \cap V_j) \setminus \bigcup \mathcal{G}_j\big) \leqslant \sigma(A_k)^{1/2} \cdot \mu(V_j \cap A_k)$$

for infinitely many $j \in \mathbb{N}$. Since $V_{j+1} \subset V_j$ and $\mu(A_k) < \infty$, we find

$$\mu\big((V \cap A_k) \setminus \bigcup \mathcal{G}\big) = \mu\Big(\bigcap_{j=1}^{\infty} V_j \cap A_k\Big) = \lim_j \mu(V_j \cap A_k) = 0\,.$$

Hence, $(V \cap A) \setminus \bigcup \mathcal{G}$ is μ negligible as the countable union of μ negligible sets. Since V was arbitrary, $\mathcal{F}$ is μ adequate for A. □

Corollary 1.7.4. Let $\mathcal{F}$ consist of closed sets, $A = \bigcup_{j=0}^{\infty} A_j$ be σ-finite. If $\mathcal{F}$ is μ adequate for each A_j, then it is μ adequate for A.

Definition 1.7.5. Let $\mathcal{F} \subset \mathcal{P}(X)$, $\delta : \mathcal{F} \to [0, R]$, and $1 < \tau < \infty$. For each $F \in \mathcal{F}$, define the (δ, τ)-*enlargement* of F as

$$\hat{F} = \hat{F}_{\delta,\tau,\mathcal{F}} = \bigcup \big\{ G \in \mathcal{F} \;\big|\; G \cap F,\, \delta(G) \leqslant \tau\delta(F) \big\}\,.$$

A common choice for δ is $\delta(F) = \operatorname{diam}(F)$. Then we have $\widehat{B(x,r)} \leqslant B\big(x, (1+2\tau)r\big)$ and $(1+2\tau)r = 5r$ for $\tau = 2$. For this reason, the following theorem is sometimes called the $5r$ covering theorem.

Theorem 1.7.6. Let $\mathcal{F} \subset \mathcal{P}(X)$, $\delta : \mathcal{F} \to]0, R]$, and $1 < \tau < \infty$. Then $\mathcal{F}$ contains a disjoint subfamily $\mathcal{G}$ such that $\bigcup \mathcal{F} \subset \bigcup \big\{ \hat{G} \;\big|\; G \in \mathcal{G} \big\}$.

We shall use the following the set theory lemma (repeatedly).

Lemma 1.7.7. Let A be set, P a reflexive predicate defined on $A \times A$, $\varrho : A \to]0, R]$, and $1 < \tau < \infty$. There exists a subset $B \subset A$ so that

$$P(a,b) \;\text{ or }\; P(b,a) \quad \text{for all } \; a, b \in B \tag{1}$$

and for all $a \in A \setminus B$,

$$\neg P(a,b) \quad \text{and} \quad \tau\varrho(b) > \varrho(a) \quad \text{for some } \; b \in B\,. \tag{2}$$

Proof. Let $\Omega \subset \mathcal{P}(A)$ consist of those $B \subset A$ such that (1) is satisfied, and for all $a \in A$, either $P(a,b)$ for all $b \in B$, or (2) is satisfied. The set Ω is ordered by inclusion. Observe $\varnothing \in \Omega$. Let $\mathcal{C} \subset \Omega$ be a maximal chain, by the Hausdorff maximality principle. Let $B = \bigcup \mathcal{C}$.

Since $\mathcal{C}$ is a chain, any two $a, b \in B$ belong to some $C \in \mathcal{C}$, thus $P(a,b)$, and condition (1) is valid. Clearly, for all $a \in A \setminus B$, either (2) is satisfied, or $P(a,b)$ for all $b \in B$. Hence $B \in \Omega$, and thus B is the largest element of $\mathcal{C}$.

Seeking a contradiction, assume that for $K = \{a' \in A | P(a',b) \text{ for all } b \in B\}$, there exists $a \in K \setminus B$. Then a may be chosen such that $\tau \cdot \varrho(a) > \sup \varrho(K)$. Let $B' = \{a\} \cup B$. Then $P(b,c)$ or $P(c,b)$ for all $b, c \in B$. Moreover, $P(a,a)$ obtains by reflexivity. For $b \in B$, we have $P(a,b)$ by assumption, so (1) is satisfied for B'. If $a' \in A \setminus B'$, then either there exists $b \in B$ such that $\neg P(a,b)$ and $\tau\varrho(b) > \varrho(a')$, or $a' \in K$. In the latter case, $\tau\varrho(a) > \varrho(a')$, so in any case, condition (2) is satisfied for B'. This proves that $B' \in \Omega$, contrary to the maximality of B. Thus, B satisfies (2) for all $a \in A \setminus B$. □

Proof of theorem 1.7.6. We may employ lemma 1.7.7 with $A = \mathcal{F}$, $\varrho = \delta$, and the predicate $P(F,G) \equiv F \neq G \Rightarrow F \cap G = \varnothing$. □

Corollary 1.7.8. Let $\mathcal{F}$ be a closed fine cover of $A \subset X$, $\delta : \mathcal{F} \to [0,R]$, and $1 < \tau < \infty$. Then there exists a disjoint $\mathcal{G} \subset \mathcal{F}$ such that $\bigcup \mathcal{F} \subset \bigcup\{\hat{G} | G \in \mathcal{G}\}$, and for any finite $\mathcal{H} \subset \mathcal{G}$,

$$A \setminus \bigcup \mathcal{H} \subset \bigcup \{\hat{G} \mid G \in \mathcal{G} \setminus \mathcal{H}\} .$$

Proof. Let $\mathcal{G} \subset \mathcal{F}$ be constructed as in theorem 1.7.6. The set $H = \bigcup \mathcal{H}$ is closed, so for any $x \in A \setminus H$, there exists $\varepsilon > 0$ such that $B(x,\varepsilon) \cap H = \varnothing$. Since $\mathcal{F}$ is fine, there exists $F \in \mathcal{F}$ with $x \in F \subset B(x,\varepsilon)$. On the other hand, by the construction of $\mathcal{G}$, there exists $G \in \mathcal{G}$ such that $F \subset \hat{G}$ and $F \cap G \neq \varnothing$. But $F \cap H = \varnothing$, so we see that $G \notin \mathcal{H}$. □

Theorem (Vitali-Federer) 1.7.9. Let μ be finite on bounded subsets, $\mathcal{F}$ a closed fine cover of A, $\delta : \mathcal{F} \to]0,R]$, and $1 < \tau, \lambda < \infty$. If μ satisfies the *doubling condition* $\mu(\hat{F}) < \lambda\mu(F)$ for all $F \in \mathcal{F}$, then $\mathcal{F}$ is μ adequate.

Proof. In particular, X is σ-finite for μ. In view of corollary 1.7.4, we may assume that A is bounded. By the same token, it suffices to consider bounded open $V \subset X$. The family $\mathcal{F} \mid V$ is a closed fine cover of $V \cap A$. Apply corollary 1.7.8 to obtain a disjoint $\mathcal{G} \subset \mathcal{F} \mid V$ such that

$$(V \cap A) \setminus \bigcup \mathcal{H} \subset \bigcup \{\hat{G} \mid G \in \mathcal{G} \setminus \mathcal{H}\}$$

for all finite $\mathcal{H} \subset \mathcal{G}$. Because $0 \leqslant \mu(\hat{G}) < \lambda \cdot \mu(G)$ for all $G \in \mathcal{G}$, the set $\mathcal{G}$ is countable. Otherwise, we would have $\mu(G) \geqslant \frac{1}{k}$ for some $k \geqslant 1$ and $G \in \mathcal{G}'$ where $\mathcal{G}'$ is some infinite countable subset set of $\mathcal{G}$, and this would imply $\mu(V) \geqslant \sum_{G \in \mathcal{G}'} \mu(G) = \infty$, contradiction. Hence,

$$\sum_{G \in \mathcal{G}} \mu(\hat{G}) \leqslant \lambda \cdot \sum_{G \in \mathcal{G}} \mu(G) \leqslant \lambda\mu(V) < \infty .$$

Moreover, for all ε, there exists a finite $\mathcal{H} \subset \mathcal{G}$ such that

$$\varepsilon \geqslant \sum_{G \in \mathcal{G} \setminus \mathcal{H}} \mu(\hat{G}) \geqslant \mu\big((V \cap A) \setminus \bigcup \mathcal{H}\big) \geqslant \mu\big((V \cap A) \setminus \bigcup \mathcal{G}\big) .$$

Therefore, the right hand side vanishes, which proves the theorem. □

1.7.10. Vitali's classical theorem is obtained from theorem 1.7.9 as follows: Let μ be finite on bounded subsets, $\mathcal{F}$ a fine family of closed balls, $\delta(B(x,r)) = \operatorname{diam}(B(x,r)) = 2r$. Whenever μ satisfies the *diametric regularity condition*

$$\mu\big(B(a,(1+2\tau)r)\big) < \lambda \cdot \mu\big(B(a,r)\big) \quad \text{for all } B(a,r) \in \mathcal{F}\,,$$

then $\mathcal{F}$ is μ adequate for any $A \subset \bigcup \mathcal{F}$. (The subfamily of $\mathcal{F}$ consisting all balls with radius $\leqslant R$ for some $R > 0$ is still fine.)

Sufficient (but not necessary) for the diametric regularity is the existence of positive numbers $0 < \alpha, \beta < \infty$ such that

$$\alpha \leqslant \frac{\mu\big(B(a,s)\big)}{s^n} \leqslant \beta \quad \text{for all } s = r, (1+2\tau)r\,,\, B(a,r) \in \mathcal{F}\,.$$

For instance, $\mathcal{L}^n(B(a,r)) = c_n \cdot r^n$ where $c_n > 0$ is some constant only depending on $n \in \mathbb{N}$, and $\mathcal{L}^n$ is n-dimensional Lebesgue measure on $\mathbb{R}^n$.

We shall have to keep track of the centres of balls in the following, and in a general metric space, these are not determined uniquely by the ball in consideration (e.g., consider a discrete space X, $\#X \geqslant 2$). Hence the following definition.

Definition 1.7.11. A *covering relation* is a subset of $\{(a,A) | a \in A \subset X\}$. For a covering relation $\mathcal{F}$ and a subset $Y \subset$, let $\mathcal{F}(V) = \operatorname{pr}_2\big((V \times \mathcal{P}(X)) \cap \mathcal{F}\big)$. Then $\mathcal{F}$ is said to be fine, open, closed, Borel etc. if $\mathcal{F}(X)$ is fine, open, closed, Borel, etc.

Let $\mathcal{V}$ be a fine Borel covering relation of X. We say that $\mathcal{V}$ is a μ *Vitali relation* if for each $Y \subset X$ and each $\mathcal{W} \subset \mathcal{V}$ which is fine on Y, $\mathcal{W}(Y)$ contains a countable disjoint subfamily μ almost covering Y.

When $x \in X$, $\mathcal{G} \subset \mathcal{P}(X)$, $f : \mathcal{G} \to \bar{\mathbb{R}}$, and $\mathcal{V}$ is a covering relation fine at x, define

$$\limsup\nolimits_{\mathcal{V}\to x} f = \limsup\nolimits_{\mathcal{V}\ni F \to x} f(F) = \lim\nolimits_{\varepsilon\to 0+} \sup\nolimits_{(x,F)\in\mathcal{F}\cap(X\times\mathcal{G})\,,\,\operatorname{diam}(F)\leqslant\varepsilon} f(F)\,,$$

and similarly for $\liminf_{\mathcal{V}\to x} f$. If $\limsup_{\mathcal{V}\to x} f = \liminf_{\mathcal{F}\to x} f$, we write $\lim_{\mathcal{V}\to x} f$ for the common value.

Theorem 1.7.12. Let μ be finite on bounded subsets, $\mathcal{V}$ a bounded, closed, and fine covering relation, $\delta : \mathcal{V}(X) \to]0,\infty]$, and $1 < \tau < \infty$. Then $\mathcal{V}$ is a μ Vitali relation whenever

$$0 \leqslant \nu(x) = \limsup\nolimits_{\mathcal{V}\ni F\to x}\Big[\delta(F) + \frac{\mu(\hat{F})}{\mu(F)}\Big] < \infty \quad \text{for } \mu\text{-a.e. } x \in X\,.$$

Proof. Let $Y \subset X$ and $\mathcal{W} \subset \mathcal{V}$ be fine on Y. Let

$$\mathcal{W}_n = \left\{ F \in \mathcal{W}(Y) \;\middle|\; \delta(F) + \frac{\mu(\hat{F})}{\mu(F)} < n \right\} \quad \text{for all } n \in \mathbb{N}\,.$$

Then $\mathcal{W}_n$ is fine on $A_n = Y \cap \{\nu < n\}$. Clearly, $\delta \mid \mathcal{W}_n < n$, and $\mu(\hat{F}) < n \cdot \mu(F)$ for all $F \in \mathcal{W}_n$, so by Vitali's theorem 1.7.9, $\mathcal{W}_n$ is μ adequate for A_n. In particular, $\mathcal{W}(Y)$ is μ adequate for A_n. Since (A_n) μ almost covers Y, corollary 1.7.4 implies that $\mathcal{W}(Y)$ is μ adequate for Y, in particular, the assertion. □

1.8 Covering Theorems of Besicovich Type

1.8.1. In what follows, let (X, d) be metric and μ a Borel measure on X.

Definition 1.8.2. Let $a, b, c \in X$, $b, c \neq a$. The *angle* $\angle(a, b, c)$ defined as

$$\angle(a,b,c) = \inf\left\{ \frac{d(x,c)}{d(a,c)} \;\middle|\; x \in X,\ d(a,x) = d(a,c),\ d(a,x) + d(x,b) = d(a,b) \right\}$$

if b, c are ordered such that $d(a,c) \leqslant d(a,b)$, and by interchanging b and c, otherwise. The condition $d(a,x) + d(x,b) = d(a,b)$ means that x lies on a geodesic from a to b, so x may be thought of a 'normalisation' of b (w.r.t. the origin a).

1.8.3. If X is some normed vector space, let $a = 0$. Then $x = \frac{\|c\|}{\|b\|} \cdot b$ and

$$\angle(0,b,c) = \left\| \frac{b}{\|b\|} - \frac{c}{\|c\|} \right\|.$$

Let X be finite-dimensional, so $\mathbb{S}(X)$ is compact. For each $\eta > 0$, there exists a number N such that $\mathbb{S}(X)$ is the union of N closed balls of radius $\frac{\eta}{2}$.

If $B \subset X \setminus 0$ such that $\angle(0,b,c) \geqslant \eta$ for all distinct $b, c \in B$, then each $\frac{b}{\|b\|}$ is contained in at most one of these balls, whence $\#B \leqslant N$. (Note that the condition $\angle(0,b,c) > 0$ implies that b, c are not collinear.) This consideration suggests the following definition.

Definition 1.8.4. Fix a triple (ξ, η, ζ) where $0 < \xi \leqslant \infty$, $0 < \eta \leqslant \frac{1}{3}$, and $1 \leqslant \zeta \in \mathbb{N}$. The metric d of X is said to be *directionally* (ξ, η, ζ)*-limited on* $A \subset X$ if the following is true: Whenever $a \in A$, then any selection $B \subset A \cap B(a, \xi)^\circ \setminus a$ such that $\angle(a, b, c) \geqslant \eta$ for all $b, c \in B$, we have $\#B \leqslant \zeta$.

1.8.5. This notion is clearly invariant under isometries. Since translations on a normed vector space are isometries and act transitively, our above considerations show that in particular, the metric induced by the norm on any finite dimensional normed vector space X is directionally (∞, η, ζ)-limited on X for any $\eta > 0$ and ζ greater or equal the Lebesgue $\frac{\eta}{2}$-number of $\mathbb{S}(X)$. The geodesic length metric of a Riemannian manifold with suitable curvature bounds is also directionally (ξ, η, ζ)-limited for some choice of parameters.

In a general metric space, the centre and the radius of a ball $B = B(x, r)$ are usually not determined uniquely by B (consider, e.g., any discrete space X, $\#X \geqslant 2$). We shall therefore consider subsets $P \subset X \times]0, \infty]$ of pairs of centres and radii before passing to the corresponding families of balls.

Definition 1.8.6. Let $P \subset X \times]0,\infty]$ and $1 < \tau < \infty$. We say that P is τ *controlled* if and only if for all $(a,r),(b,s) \in P$, $(a,r) \neq (b,s)$,

$$d(a,b) > r > \frac{s}{\tau} \quad \text{or} \quad d(a,b) > s > \frac{r}{\tau}\,.$$

In particular, $a \neq b$ and $a \notin B(b,s)$ or $b \notin B(a,r)$, so $B(a,r) \neq B(b,s)$. Thus, for the family of closed balls associated to a τ controlled set P, the centre of each $B \in P$ is uniquely determined.

Given (ξ,η,ζ), a number τ shall be termed *permissible* if

$$1 < \tau < 2-\eta \quad \text{and} \quad \eta + \frac{\tau}{2-\eta} + \tau(\tau-1) < 1\,.$$

Since $\eta \leqslant \frac{1}{3}$, so $\eta + \frac{1}{2-\eta} \leqslant \frac{1}{3} + \frac{2}{5} < 1$, any sufficiently small $\tau > 1$ is permissible.

Lemma 1.8.7. Let $1 < \tau < \infty$, $0 < \mu < \infty$, and $P \subset X \times]0,\mu]$. Then there exists $Q \subset P$ such that $d(a,b) > r+s$ for all distinct $(a,r),(b,s) \in Q$, and for all $(a,r) \in P$, there exists $(b,s) \in Q$ so that $d(a,b) \leqslant r+s$ and $s > \frac{r}{\tau}$.

Proof. We may employ lemma 1.7.7 with $A = P$, $\varrho = \mathrm{pr}_2$, and

$$P\big((a,r),(b,s)\big) \;\equiv\; (a,r) \neq (b,s) \;\Rightarrow\; d(a,b) > r+s$$

to achieve the statement. □

Proposition 1.8.8. Suppose d is directionally (ξ,η,ζ)-limited on $A \subset X$, and τ is permissible. Suppose that $P \subset A \times]0,\infty]$ is τ controlled. If there is $(a,r) \in P$ such that

$$d(a,b) < \xi\,,\; d(a,b) \leqslant r+s \quad \text{for all } (b,s) \in P \text{ such that } s > \frac{r}{\tau}\,,$$

then $\#P \leqslant 2\zeta + 1$.

Proof. Let $k = \frac{2-\eta}{\tau}$. Define

$$P_1 = \big\{(b,s) \in P \,\big|\, 0 < d(a,b) \leqslant kr\big\} \quad \text{and} \quad P_2 = \big\{(b,s) \in P \,\big|\, d(a,b) > kr\big\}\,.$$

We have already noted that since P is τ controlled, $\mathrm{pr}_1 : P_j \to B_j$ is a bijection onto $B_j = \mathrm{pr}_j(P_j)$. Thus $\#P - 1 = \#B_1 + \#B_2$. We shall prove that any two disctinct $b,c \in B_j$ satisfy $\angle(a,b,c) \geqslant \eta$, to conclude $\#B_j \leqslant \zeta$, whence the theorem.

Let $(b,s),(c,t) \in P_j$ be distinct, $d(a,b) \geqslant d(a,c)$. Thus, choose any $x \in X$ such that $d(a,x) = d(a,c)$ and $d(a,x) + d(x,b) = d(a,b)$. We have

$$d(x,c) \geqslant -d(x,b) + d(b,c) = d(a,c) - d(a,b) + d(b,c)\,.$$

Now, $d(a,b) > \frac{r}{\tau}$, since $r > \frac{r}{\tau}$ and either $d(a,b) > r$ or $d(a,b) > s > \frac{r}{\tau}$, since P is τ controlled.

In case $j = 1$, we have $d(a,b) \leqslant kr$. Moreover, $d(b,c) > \frac{r}{\tau}$ because either $d(b,c) > s$ or $d(b,c) > t$, and $s,t > \frac{r}{\tau}$ by assumption. Hence,

$$d(x,c) > d(a,c) - kr + \frac{r}{\tau} = d(a,c) - \frac{r}{\tau} \cdot (1-\eta) \geqslant d(a,c) - d(a,c) \cdot (1-\eta) = d(a,c) \cdot \eta \,,$$

whence $\angle(a,b,c) \geqslant \eta$.

In case $j = 2$, $d(a,b) \leqslant r + s$, and $d(a,c) > \min\big(kr, \frac{t}{\tau}\big)$. Moreover, $d(b,c) > s$ or $d(b,c) > t > \frac{s}{\tau}$, so $d(b,c) - s > 0 > (1-\tau)t$ or

$$d(b,c) - s > t - s > t - \tau t = (1-\tau)t \,.$$

In any case, $d(b,c) - s > (1-\tau)t$, which implies

$$\begin{aligned} d(x,c) &> d(a,c) - r + d(b,c) - s > d(a,c)(1-k^{-1}) - t(\tau-1) \\ &> d(a,c) \cdot \Big(1 - \frac{\tau}{2-\eta} - \tau(\tau-1)\Big) > d(a,c) \cdot \eta \,, \end{aligned}$$

because τ is permissible. Hence, $\angle(a,b,c) \geqslant \eta$. This completes the proof. □

Proposition 1.8.9. Let d be directionally (ξ, η, ζ) limited, τ be permissible, and assume the subset $P \subset X \times]0,\mu[$, where $0 < \mu < \frac{\xi}{2}$, is τ controlled. Then, for the covering relation

$$B_P = \big\{ \big(a, B(a,r)\big) \bigm| (a,r) \in P \big\} \,,$$

$B_P(X)$ is the union of $2\zeta + 1$ disjoint subfamilies.

Proof. Let $P_0 = P$, $Q_0 = \varnothing$, define $P_j = P_{j-1} \setminus Q_{j-1}$, and let $Q_j \subset P_j$ be subset constructed in lemma 1.8.7. Since $d(a,b) > r + s$ for all distinct $(a,r), (b,s) \in Q_j$, the collection $B_{Q_j}(X)$ is disjoint. The proof shall be complete once we have shown that $P_{2\zeta+2} = \varnothing$.

To that end, seeking a contradiction, assume the existence of $(a,r) \in P_{2\zeta+2} \subset P_j$. Then there exist $(b_j, s_j) \in Q_j$ such that

$$d(a,b_j) \leqslant r + s_j < \xi \quad \text{and} \quad s_j > \frac{r}{\tau} \,.$$

The collection $Q = \{(a,r)\} \cup \{(b_j,s_j) | j = 1, \dots, 2\zeta+1\}$ has $\#Q = 2\zeta + 2$. On the other hand, it is τ controlled as a subset of P, and hence satisfies the assumptions of proposition 1.8.8, so $\#Q \leqslant 2\zeta + 1$, contradiction. □

Lemma 1.8.10. If $1 < \tau < \infty$, $0 < \mu < \infty$, and $P \subset X \times]0,\mu[$, then there exists a τ controlled $Q \subset P$ such that $\mathrm{pr}_1(P) \subset \bigcup B_Q(X)$.

Proof. Apply lemma 1.7.7 with $A = P$, $\varrho(a,r) = r$,

$$P\big((a,r),(b,s)\big) \;\equiv\; (a,r) \neq (b,s) \;\Rightarrow\; d(a,b) > s > \frac{r}{\tau} \,.$$

Thus there exists $R \subset P$ such that $P\big((a,r),(b,s)\big)$ or $P\big((b,s),(a,r)\big)$ for all $(a,r),(b,s) \in$

R, which means that R is τ controlled. Moreover, for all $(a,r) \in P$, there exists $(b,s) \in R$ such that $d(a,b) \leqslant s$ or $s \leqslant \frac{r}{\tau}$, and $\tau s > r$; thus, $d(a,b) \leqslant s$, and this means $a \in B(b,s) \in B_R(X)$. Hence the assertion. □

The essence of the following theorem is that under favourable conditions (i.e. if the metric is directionally limited), there is a fixed number N such that ball covers with self-intersection number bounded by $2N$ suffice to cover any subset.

Besicovich Covering Theorem 1.8.11. Let d be directionally (ξ,η,ζ) limited on $A \subset X$, and fix $0 < \mu < \frac{\xi}{2}$. If $\mathcal{F} = B_P(X)$ for some $P \subset X \times]0,\mu[$, $A \subset \operatorname{pr}_1(P)$, i.e. $\mathcal{F}$ is a collection of closed balls with radii less than μ, and such that any $a \in A$ is the centre of some ball in $\mathcal{F}$, then A is contained in the union of $2\zeta + 1$ disjoint subfamilies of $\mathcal{F}$.

Proof. First choose some permissible τ. There exists, by lemma 1.8.10, a τ controlled subset $Q \subset P$, such that $A \subset \operatorname{pr}_1(P) \subset \bigcup B_Q(X)$. By proposition 1.8.9, $B_Q(X)$ is the union of $2\zeta + 1$ disjoint subfamilies. □

Corollary 1.8.12. Under the conditions of theorem 1.8.11, assume that μ is a Borel measure on X and A is separable and σ-finite for μ. Then $\mathcal{F}$ is μ adequate for A if it is fine. Moreover, the σ-finiteness of A is immediate if μ is finite on bounded subsets.

Proof. We show that the hypothesis of theorem 1.7.3 is fulfilled for the choices $A_k = A$ and $\sigma(A) = \frac{2\zeta}{2\zeta+1}$.

Let $V \subset X$ be open. Then any point of $V \cap A$ is the centre of some member of $\mathcal{F} \mid V$. Therefore, theorem 1.8.11 gives the existence of disjoint subfamilies $\mathcal{H}_j \subset \mathcal{F} \mid V$, $j = 1,\dots,2\zeta+1$, such that $V \cap A \subset \bigcup_{j=1}^{2\zeta+1} \bigcup \mathcal{H}_j$. Since A is separable, we may assume each of the $\mathcal{H}_j$ be countable. In particular, if μ is finite on bounded subsets, the σ-finiteness for μ of A follows. In any case, $\bigcup \mathcal{H}_j$ is μ measurable.

Observe

$$\mu(V \cap A) \leqslant \sum_{j=1}^{2\zeta+1} \mu\big(A \cap \bigcup \mathcal{H}_j\big) .$$

Hence, there exists $1 \leqslant j \leqslant 2\zeta + 1$ such that $\mu\big(A \cap \bigcup \mathcal{H}_j\big) \geqslant \frac{1}{2\zeta+1} \cdot \mu(V \cap A)$. By the measurability of $\bigcup \mathcal{H}_j$, we infer

$$\mu\big((V \cap A) \setminus \bigcup \mathcal{H}_j\big) = \mu(V \cap A) - \mu\big(A \setminus \bigcup \mathcal{H}_j\big) \leqslant \frac{2\zeta}{2\zeta - 1} \cdot \mu(V \cap A) ,$$

which is just the required relation. □

Corollary 1.8.13. Assume $X = \bigcup_{k=0}^{\infty} A_k$ where d is directionally limited on A_k for all k. If X is separable and σ-finite for μ (e.g. μ is finite on bounded subsets), then $\mathcal{V} = B_{X \times]0,\infty[}$, the covering relation of all closed balls in X, is a μ Vitali relation.

Proof. Let $\mathcal{W} \subset \mathcal{V}$, $Y \subset X$, such that $\mathcal{W}$ is fine on Y. By corollary 1.8.12, $\mathcal{W}(Y)$ is μ adequate for $Y \cap A_k$ for all $k \in \mathbb{N}$. By corollary 1.7.4, $\mathcal{W}(Y)$ is μ adequate for Y. □

1.9 Differentiation of Measures

In all that follows, let (X, d) be separable and the countable union of sets on which d is directionally limited. Moreover, let μ, λ be Borel regular measures such that λ is finite on bounded subsets. By corollary 1.8.13, $\mathcal{V} = B_{X\times]0,\infty[}$ is a Vitali relation for λ .

Definition 1.9.1. Let $x \in X$. Define $\frac{r}{0} = \infty$ for all $\infty \geqslant r > 0$, $\frac{0}{r} = 0$ for all $r \in [0, \infty]$, so that $\frac{a}{b} \leqslant c \;\Leftrightarrow\; a \leqslant bc$ for all $a, b \in [0, \infty]$, $c \in [0, \infty[$; let

$$D^+_\lambda \mu(x) = \limsup_{\varepsilon\to 0+} \frac{\mu\big(B(x,\varepsilon)\big)}{\lambda\big(B(x,\varepsilon)\big)} \quad \text{and} \quad D^-_\lambda \mu(x) = \liminf_{\varepsilon\to 0+} \frac{\mu\big(B(x,\varepsilon)\big)}{\lambda\big(B(x,\varepsilon)\big)} \,.$$

Whenever $D^\pm_\lambda \mu(x)$ are equal, let $\frac{d\mu}{d\lambda}(x) = D_\lambda \mu(x) = D^\pm_\lambda \mu(x)$. Whenever $D_\lambda \mu$ exists, it is called the Radon-Nikodým derivative of μ by λ , or the density of μ w.r.t. λ .

The aim of this subsection is to establish sufficient and necessary conditions for the existence of $D_\lambda \mu$, and to see how μ can be recovered by integrating $D_\lambda \mu$ against λ .

Lemma 1.9.2. Let $0 < \alpha < \infty$ and $A \subset X$. Then

(i). $A \subset \{D^-_\lambda \mu \leqslant \alpha\}$ implies $\mu(A) \leqslant \alpha \cdot \lambda(A)$, and

(ii). $A \subset \{D^+_\lambda \mu \geqslant \alpha\}$ implies $\mu(A) \geqslant \alpha \cdot \lambda(A)$.

Proof. Since A is σ-finite for λ , we may assume $\lambda(A) < \infty$. Let $\varepsilon > 0$. By theorem 1.1.9 (i), there is an open $U \supset A$ such that $\lambda(U) \leqslant \lambda(A) + \varepsilon$.

For the subfamily $\mathcal{W} \subset \mathcal{V}$ of all (a, B) , $B = B(a, r) \subset U$, for which $a \in A$ and $\mu(B) \leqslant (\alpha + \varepsilon)\lambda(B)$, $\mathcal{W}$ is fine on A by assumption. Also by assumption, A is contained in the union of balls of finite μ measure, so A is σ-finite for μ . By corollary 1.8.13, $\mathcal{W}$ is μ Vitali relation on A . Thus, $\mathcal{W}(A)$ is μ adequate for A , and there exists a countable disjoint $\mathcal{G} \subset \mathcal{W}(A)$ μ almost covering A . Then,

$$\mu(A) \leqslant \sum_{B\in\mathcal{G}} \mu(B) \leqslant (\alpha+\varepsilon) \cdot \sum_{B\in\mathcal{G}} \lambda(B) \leqslant (\alpha+\varepsilon)\cdot\lambda(U) \leqslant (\alpha+\varepsilon)\cdot\big(\lambda(A)+\varepsilon\big) \,.$$

Hence follows the assertion (i). The statement (ii) follows analogously. □

Proposition 1.9.3. The function $D_\lambda \mu$ (defined as ∞ whenever $D_\lambda \mu(x)$ does not exist) is λ measurable. Moreover, $D_\lambda \mu(x)$ exists and is finite for λ a.e. $x \in X$.

Proof. First, let us establish the λ measurability. To that end, fix $0 < a < b < \infty$. It suffices to prove

$$\lambda(C) \geqslant \lambda\big(C \cap \{D_\lambda \mu < a\}\big) + \lambda\big(C \cap \{D_\lambda \mu \geqslant b\}\big) \quad \text{for all } C \subset X \,.$$

Indeed, this implies by theorem 1.1.10 that $\mu \llcorner \big(D_\lambda \mu^{-1}(\bar{\mathbb{R}})\big)$ is a Borel measure.

Thus, let $A \subset \{D_\lambda \mu < a\}$ and $B \subset \{D_\lambda \mu \geqslant b\}$ be bounded. There exist Borel sets $A_0, A_1 \supset A$ such that $\lambda(A) = \lambda(A_0)$ and $\mu(A) = \mu(A_1)$. Then $A \subset A_0 \cap A_1 \subset A_j$,

$j = 0, 1$, so $\lambda(A) = \lambda(A')$ and $\mu(A) = \mu(A')$ for the Borel set $A' = A_0 \cap A_1$. Similarly, there is a Borel $B' \subset B$ such that $\lambda(B) = \lambda(B')$, $\mu(B) = \mu(B')$.

Then

$$u(A' \cap B') \geqslant \mu(A \cap B') = \mu(A') - \mu(A \setminus B') \geqslant \mu(A' \cap B') ,$$

so $\mu(A' \cap B') = \mu(A \cap B') = \mu(A' \cap B)$, and equally for λ. By lemma 1.9.2,

$$a\lambda(A' \cap B') = a\lambda(A \cap B') \geqslant \mu(A \cap B') = \mu(A' \cap B) \geqslant b\lambda(A' \cap B) = b\lambda(A' \cap B') .$$

Since $0 < a < b < \infty$, we find $\lambda(A' \cap B') = 0$. Finally,

$$\lambda(A \cup B) = \lambda\big((A \cup B) \cap A'\big) + \lambda\big((A \cup B) \cap B'\big) \geqslant \lambda(A) + \lambda(B) ,$$

and this proves the λ measurability of $D_\lambda \mu$.

To see that $D_\lambda \mu$ exists λ a.e., consider the subfamily $\mathcal{W} \subset \mathcal{V}$ consisting of the pairs $(a, B(a,r))$ with $B(a,r)$ λ negligible. Then the set of points x at which $D_\lambda \mu(x)$ does not exist or is infinite is the following union:

$$P \cup Q \cup \bigcup_{a,b \in \mathbb{Q}, a<b} \{D_\lambda^- \mu < a < b < D_\lambda^+ \mu\} .$$

Here, P is the set of points at which $\mathcal{W}$ is fine, and Q the set of all x so that $D_\lambda^+ \mu(x) = \infty$. Since $\mathcal{W}(P)$ is fine on P, it is λ adequate for P, and hence P is λ negligible.

For any bounded $A \subset Q$, lemma 1.9.2 gives $c\lambda(A) \leqslant \mu(A) < \infty$ for all $c > 0$, so $\lambda(A) = 0$, and Q is λ negligible. For any bounded $A \subset \{D_\lambda^- < a < b < D_\lambda^+ \mu\}$, the lemma gives $b\lambda(A) \leqslant \mu(A) \leqslant a\lambda(A)$, so $\lambda(A) = 0$ again. In conclusion, the set of all points x for which $D_\lambda \mu(x)$ does not exist or is infinite is a λ zero set. □

Definition 1.9.4. We say that μ is *absolutely continuous* w.r.t. λ, in symbols, $\mu \ll \lambda$, if $\lambda(A) = 0$ implies $\mu(A) = 0$ for all $A \subset X$. If, on the other hand, there exists a Borel $B \subset X$ such that $\lambda(X \setminus B) = \mu(B) = 0$, then λ and μ are said to be *mutually singular*, written $\mu \perp \lambda$. Obviously, the relation $\ll$ is a quasi-order (reflexive and transitive), and $\perp$ is symmetric.

Theorem (Radon-Nikodým) 1.9.5. We have

$$\mu(B) \geqslant \int_B D_\lambda \mu \, d\lambda \quad \text{for all } B \in \mathcal{B}(X) .$$

Moreover, we have equality if and only if $\mu \ll \lambda$, if and only if $D_\lambda^- \mu < \infty$ μ a.e.

Proof. Let $1 < t < \infty$. Since $D_\lambda \mu$ is λ a.e. existent and finite by proposition 1.9.3 and corollary 1.5.6, we have

$$\int_B D_\lambda \mu \, d\lambda = \sum_{k=-\infty}^{\infty} \int_{B \cap \{t^k \leqslant D_\lambda \mu < t^{k+1}\}} D_\lambda \mu \, d\lambda \leqslant \sum_{k=-\infty}^{\infty} t^{k+1} \cdot \lambda\big(B \cap \{t^k \leqslant D_\lambda \mu < t^{k+1}\}\big)$$

$$\leqslant t \cdot \sum_{k=-\infty}^{\infty} \mu\big(B \cap \{t^k \leqslant D_\lambda \mu < t^{k+1}\}\big) \leqslant t \cdot \mu(B)\,,$$

which proves the inequality.

By proposition 1.9.3, $D_\lambda \mu < \infty$ λ-a.e. If $\mu \ll \lambda$, the statement holds μ a.e. Hence,

$$\begin{aligned}\mu(B) &= \sum_{k=-\infty}^{\infty} \mu\big(B \cap \{t^k \leqslant D_\lambda \mu < t^{k+1}\}\big) \leqslant \sum_{k=-\infty}^{\infty} t^{k+1} \lambda\big(B \cap \{t^k \leqslant D_\lambda \mu < t^{k+1}\}\big) \\ &\leqslant t \cdot \sum_{k=-\infty}^{\infty} \int_{\{t^k \leqslant D_\lambda \mu < t^{k+1}\}} D_\lambda \mu \, d\lambda \leqslant t \cdot \int_B D_\lambda \mu \, d\lambda\,.\end{aligned}$$

If, conversely, the equality holds for all Borel $B \subset X$, let $A \subset X$ be λ negligible. There exists a Borel $B \supset A$ such that $\lambda(B) = \lambda(A)$. Then $\mu(A) \leqslant \mu(B) = \int_B D_\lambda \mu \, d\lambda = 0$, so we have $\mu \ll \lambda$.

If $\mu \ll \lambda$, note that $D_\lambda^- \mu \leqslant D_\lambda \mu < \infty$ λ a.e. implies finiteness μ a.e.

Finally, assume $D_\mu^- \lambda < \infty$ μ a.e. Let $A \subset X$, $\lambda(A) = 0$. For any $n \in \mathbb{N}$,

$$\mu\big(A \cap \{D_\lambda^- \mu \leqslant n\}\big) \leqslant n\lambda(A) = 0\,,$$

so $\mu(A) \leqslant \sum_{n=0}^{\infty} \mu(A \cap \{D_\lambda^- \mu \leqslant n\}) = 0$. □

Corollary 1.9.6. If λ and μ coincide on all small closed balls, they are equal.

Proof. Indeed, if this is the case, then $D_\mu \lambda(x) = 1$ for all $x \in X$, so by theorem 1.9.5,

$$\mu(B) = \int_B D_\lambda \mu \, d\lambda = \lambda(B) \quad \text{for all } \; B \in \mathcal{B}(X)\,.$$

Since μ and λ are Borel regular, this gives $\mu = \lambda$ by the device used in the proof of measurability in proposition 1.9.3. □

Remark 1.9.7. The above statement is false in arbitrary metric spaces, even if they are supposed to be compact. (Recall that we have assumed the condition that the metric is σ-directionally limited.)

Lebesgue Decomposition Theorem 1.9.8. There exists a Borel regular measure $\nu \leqslant \mu$, finite on bounded subsets, and a λ measurable $f : X \to [0, \infty]$, such that $\lambda \perp \nu$ and

$$\mu(B) = \int_B f \, d\lambda + \nu(B) \quad \text{for all } \; B \in \mathcal{B}(X)\,.$$

Moreover, $\nu = 0$ if and only if $\mu \ll \lambda$.

Proof. Let $A = \{D_\lambda^- \mu = \infty\}$, $\nu = \mu \llcorner A$. By proposition 1.9.3, $\lambda(A) = 0$, so that $\lambda \perp \nu$. Since $\mu \llcorner (X \setminus A)(A) = 0$ and $D_\lambda^-(\mu \llcorner (X \setminus A)) \leqslant D_\lambda^- \mu$, we have $D_\lambda^-(\mu \llcorner (X \setminus A))$ is finite $\mu \llcorner (X \setminus A)$-a.e. By theorem 1.9.5, this implies $\mu - \nu \ll \lambda$. By the same token,

$$\mu(B) = \big(\mu \llcorner (X \setminus A)\big)(B) + \nu(B) = \int_B D_\lambda\big(\mu \llcorner (X \setminus A)\big) \, d\lambda + \nu(B) \text{ for all } \; B \in \mathcal{B}(X)\,.$$

Finally, $\mu \ll \lambda$ if and only if $\mu(A) = 0$, if and only if $\nu = \mu \llcorner A = 0$. ∎

Lebesgue-Besicovich Differentiation Theorem 1.9.9. Let $f : X \to \bar{\mathbb{R}}$ be a locally λ summable function. Then

$$\lim_{\varepsilon \to 0+} \frac{1}{\lambda(B(x,\varepsilon))} \int_{B(x,\varepsilon)} f \, d\lambda = f(x) \quad \text{for } \lambda \text{ a.e. } x \in X .$$

The function f^* which equals the left hand side whenever the limit exists, and zero otherwise, is called the *precise representative* of f.

Proof. We may assume $f \geqslant 0$. Define μ by $\mu(A) = \int_A f \, d\lambda$ for all $A \subset X$. Then μ is Borel regular and finite on bounded subsets. Moreover, $\mu \ll \lambda$, so

$$\int_B D_\lambda \mu \, d\lambda = \mu(B) = \int_B f \, d\lambda \quad \text{for all } B \in \mathcal{B}(X) .$$

This implies that $D_\lambda \mu = f$ λ a.e. Thus,

$$\frac{1}{\lambda\big(B(x,\varepsilon)\big)} \cdot \int_{B(x,\varepsilon)} f \, d\lambda = \frac{\mu\big(B(x,\varepsilon)\big)}{\lambda\big(B(x,\varepsilon)\big)}$$

converges to $D_\lambda \mu(x) = f(x)$ for λ a.e. $x \in X$, by proposition 1.9.3. ∎

Corollary 1.9.10. If $A \subset X$ is λ measurable, then

$$\lim_{\varepsilon \to 0+} \frac{\lambda\big(A \cap B(x,\varepsilon)\big)}{\lambda\big(B(x,\varepsilon)\big)} = 1_A(x) \quad \text{for } \lambda \text{ a.e. } x \in X .$$

Corresponding to whether the left hand side is 0 or 1 for $x \in X$, this point is called *of λ density* 0 *resp.* 1 *for A.*

Proof. This is just theorem 1.9.9, applied to the function 1_A. ∎

Definition 1.9.11. Let $1 \leqslant p < \infty$ and $f : X \to \bar{\mathbb{R}}$. Then f is said to be *p-summable for μ* if f is μ measurable and $|f|^p$ is μ-summable. f is said to be *locally p-summable for μ* if f is μ-measurable and p-summable for $\mu \llcorner U$ for U in a neighbourhood basis for X.

Corollary 1.9.12. Let $1 \leqslant p < \infty$ and $f : X \to \bar{\mathbb{R}}$ be locally p-summable for μ. Then

$$\lim_{\varepsilon \to 0+} \frac{1}{\lambda\big(B(x,\varepsilon)\big)} \cdot \int_{B(x,\varepsilon)} |f - f(x)|^p \, d\lambda = 0 \quad \text{for } \lambda \text{ a.e. } x \in X .$$

A point $x \in X$ for which this equality holds is called a *Lebesgue point* of f.

Proof. Let $(r_k) \subset \mathbb{R}$ be a dense sequence. By theorem 1.9.9, there exists a μ cozero $A \subset X$ such that

$$\lim_{\varepsilon \to 0+} \frac{1}{\lambda\big(B(x,\varepsilon)\big)} \cdot \int_{B(x,\varepsilon)} |f - r_k|^p \, d\lambda = |f(x) - r_k|^p \quad \text{for all } x \in A , \, k \in \mathbb{N} .$$

Let $x \in A$ and $\delta > 0$, and fix some integer $k \in \mathbb{N}$ such that $|f(x) - r_k|^p \leqslant \frac{\delta}{2^p}$. Abbreviate $\gamma_\varepsilon = \lambda\big(B(x,\varepsilon)\big)^{-1}$. We may apply the inequality

$$(a+b)^p \leqslant 2^{p-1} \cdot (a^p + b^p) \quad \text{valid for all} \quad a, b \geqslant 0$$

to the effect that

$$\begin{aligned}
&\limsup_{\varepsilon \to 0+} \gamma_\varepsilon \int_{B(x,\varepsilon)} |f - f(x)|^p \, d\lambda \\
&\qquad \leqslant 2^{p-1} \cdot \Big[\limsup_{\varepsilon \to 0+} \gamma_\varepsilon \int_{B(x,\varepsilon)} |f - r_k|^p \, d\lambda + |f(x) - r_k|^p\Big] \\
&\qquad = 2^p \cdot |f(x) - r_k|^p \leqslant \delta \, ,
\end{aligned}$$

whence our claim. ☐

The following corollary is established by the same principle.

Corollary 1.9.13. Let $f : X \to E$ be μ measurable, where E is a separable normed vector space. If $\|f\|$ is $\mu \llcorner A$ summable for each bounded $A \in \mathfrak{M}(\mu)$, then

$$\lim_{\varepsilon \to 0+} \frac{1}{\lambda\big(B(x,r)\big)} \cdot \int \|f - f(x)\| \, d\lambda = 0 \quad \text{for } \lambda \text{ a.e. } x \in X \, .$$

Proof. For $x \in X$, Let A_x be the set of all $e \in E$ such that

$$\lim_{\varepsilon \to 0+} \frac{1}{\lambda\big(B(x,r)\big)} \cdot \int \|f - e\| \, d\lambda = \|f(x) - e\| \, .$$

Fix a dense sequence $(e_k) \subset E$, and apply theorem 1.9.9 to $\|f - e_k\|$ to achieve the existence of a λ negligible set N_k so that $e_k \in A_x$ for all $x \in X \setminus N_k$. Letting $N = \bigcup_{k=0}^{\infty} A_k$, we find $e_k \in A_x$ for all $k \in \mathbb{N}$, $x \in X \setminus N$. By the same device as above, $A_x = X$ for all $x \in X \setminus N$, so the assertion follows. ☐

Remark 1.9.14. Our presentation follows [EG92] rather closely, although some parts are from [Mat95] (in particular some of the material on differentiation of measures), and others from [Fed69] (in particular the subsections on covering theorems).

2 Hausdorff Measure

2.1 Carathéodory's Construction

In all that follows, let (X,d) be metric.

Definition 2.1.1. Let $\mathcal{F} \subset \mathcal{P}(X)$, $\varrho : \mathcal{F} \to [0,\infty]$, and $0 < \delta \leqslant \infty$. Define the *size δ approximating measure μ_δ* by

$$\mu_\delta(A) = \inf\Bigl\{\textstyle\sum_{G\in\mathcal{G}} \varrho(G) \Bigm| \mathcal{G} \subset \mathcal{F} \text{ countable}, \operatorname{diam} G \leqslant \delta \text{ for all } G \in \mathcal{G}, A \subset \bigcup \mathcal{G}\Bigr\}$$

for all $A \subset X$. Then $\delta \mapsto \mu_\delta(A)$ is decreasing for all $A \subset X$, so we may define

$$\mu(A) = \lim\nolimits_{\delta\to 0+} \mu_\delta(A) \quad \text{for all } A \subset X .$$

This is the *measure associated to the Carathéodory construction for $\mathcal{F}$ and ϱ*. Clearly, it suffices to know the values of ϱ on non-void members of $\mathcal{F}$ to define μ_δ and μ.

Proposition 2.1.2. The set functions μ_δ, μ are measures. If $\mathcal{F}$ is a Borel family, μ is a Borel regular measure.

Proof. The empty set $\varnothing$ can be covered by an empty partition, so $\mu_\delta(\varnothing) = 0$. Take some $A \subset \bigcup_{k=0}^\infty A_k$. Fix $\varepsilon > 0$ and let $\mathcal{G}_k \subset \mathcal{F}$ be countable covers of A_k such that $\operatorname{diam} G \leqslant \delta$ for all $G \in \mathcal{G} = \bigcup_{k=0}^\infty \mathcal{G}_k$, and $\sum_{G\in\mathcal{G}_k} \varrho(G) \leqslant \mu_\delta(A_k) + \frac{\varepsilon}{2^{k+1}}$ for all $k \in \mathbb{N}$. Then $\mathcal{G} \subset \mathcal{F}$, and is countable. Moreover,

$$\mu_\delta(A) \leqslant \sum_{k=0}^\infty \sum_{G\in\mathcal{G}_k} \varrho(G) \leqslant \sum_{k=0}^\infty \Bigl(\mu_\delta(A_k) + \frac{\varepsilon}{2^{k+1}}\Bigr) = \sum_{k=0}^\infty \mu_\delta(A_k) + \varepsilon .$$

Hence, μ_δ is a measure. It follows easily that μ is also a measure.

Now, assume that $\mathcal{F}$ is a Borel family. Fix $A, B \subset X$ such that $\operatorname{dist}(A,B) > \delta$. Let $\mathcal{G} \subset \mathcal{F}$ be a countable cover of $A \cup B$, $\operatorname{diam} G \leqslant \delta$ for all $G \in \mathcal{G}$. Consider

$$\mathcal{H}_C = \bigl\{G \in \mathcal{G} \bigm| G \cap C \neq \varnothing\bigr\} \quad \text{for} \quad C = A, B .$$

Then $\mathcal{H}_A \cap \mathcal{H}_B = \varnothing$ and $\mathcal{H}_A \cup \mathcal{H}_B$ covers $A \cup B$ since this is the case of $\mathcal{G}$. Hence $\mathcal{H}_C$ covers C for $C = A, B$, and it follows that

$$\mu_\delta(A \cup B) \geqslant \mu_\delta(A) + \mu_\delta(B) .$$

Hence, $\mu(A \cup B) \geqslant \mu(A) + \mu(B)$ for all $A, B \subset X$ such that $\operatorname{dist}(A,B) > 0$. By Carathéodory's criterion (theorem 1.1.10), μ is a Borel measure.

Let $A \subset X$. To prove Borel regularity of μ, we may w.l.o.g. assume $\mu(A) < \infty$. Thus, there exist Borel $B_k \supset A$ such that $\mu_\delta(B_k) \leqslant \mu_\delta(A) + \frac{1}{k}$. We may assume $B_{k+1} \leqslant B_k$, and thus

$$\mu_\delta(A) \leqslant \mu_\delta(B) = \lim_k \mu_\delta(B_k) \leqslant \mu_\delta(A)$$

for $B = \bigcap_{j=0}^{\infty} B_j$. Thus, there exist Borel $C_k \supset A$ such that $\mu_{1/k}(A) = \mu_{1/k}(C_k)$ for all $k \geqslant 1$. Let $D_k = \bigcap_{\ell=k}^{\infty} C_\ell$. Then D_k is Borel and for any $\ell \geqslant k$,

$$\mu_{1/k}(A) \leqslant \mu_{1/k}(D_\ell) \leqslant \mu_{1/k}(C_k) = \mu_{1/k}(A)\,,$$

so $\mu_{1/k}(A) = \mu_{1/k}(D_\ell)$ for all $\ell \geqslant k$. Hence,

$$\mu(A) = \sup_k \mu_{1/k}(A) = \sup_k \inf_\ell \mu_{1/k}(D_\ell) = \inf_\ell \mu(D_\ell) = \mu(D)$$

where $D = \bigcap_{\ell=1}^{\infty} D_\ell$. □

Remark 2.1.3. The above proof follows [Fed69], with some details filled in.

2.2 Hausdorff Measure and Dimension

Definition 2.2.1. Let $0 \leqslant s < \infty$ and define $\omega_s = \pi^{s/2} \cdot \Gamma\big(\frac{s}{2}+1\big)^{-1}$. (For integer values of s, this is the Lebesgue measure of the s-dimensional Euclidean unit ball.) Set

$$\varrho_s(F) = \frac{\omega_s}{2^s} \cdot (\operatorname{diam} F)^s \quad \text{for all } \varnothing \neq F \subset X\,.$$

The *s-dimensional Hausdorff measure* $\mathcal{H}^s = \mathcal{H}^s_d$ is the measure associated to the Carathéodory construction for $\mathcal{P}(X)$ and ϱ_s defined as above. Since $\varrho_s(A) = \varrho_s(\overline{A})$, one obtains the same measure by considering instead all closed subsets of X. Hence, by proposition 2.1.2, $\mathcal{H}^s$ is a Borel regular measure. Note that it is, in general, not σ-finite.

Theorem 2.2.2. Let X be separable, Y metric, $f : X \to Y$ be Lipschitz, and $0 \leqslant s < \infty$. For any Borel $A \subset X$, we have

$$\operatorname{Lip}(f)^s \cdot \mathcal{H}^s(A) \geqslant \int N(f \mid A, y)\, d\mathcal{H}^s(y)\,.$$

Here the *multiplicity function of* f, $N(f \mid A, y) = \#A \cap f^{-1}(y)$, is $\mathcal{H}^s$ measurable.

Proof. Let $\sigma(A) = \mathcal{H}^s\big(f(A)\big)$. Since $\operatorname{diam}\big(f(F)\big) \leqslant \operatorname{Lip}(f) \cdot \operatorname{diam}(F)$, we find

$$\sigma(A) \leqslant \operatorname{Lip}(f)^s \cdot \mathcal{H}^s(A)\,.$$

Let μ be the measure associated to Carathéodory's construction for $\mathcal{B}(X)$ and σ. We wish to prove

$$\mu(A) = \int N(f \mid A, y)\, d\mathcal{H}^s(y) \quad \text{for all } A \in \mathcal{B}(X)\,.$$

Let $P_j \subset \mathcal{B}(X)$ be countable Borel partitions of A, such that each $B \in P_{j+1}$ is the union of members of P_j, and $\lim_j \sup_{B\in P_j} \operatorname{diam}(B) = 0$. Define

$$g_j = \sum_{B\in P_j} 1_{f(B)} \quad \text{for all } j \in \mathbb{N} .$$

Then, for all $y \in Y$,

$$g_j(y) = \#\{B \in P_j \mid f^{-1}(y) \cap P_j \neq \varnothing\} \leqslant N(f \mid A, y) .$$

Clearly, $g_j \leqslant g_{j+1}$. Let $B \subset f^{-1}(y) \cap A$ be a finite subset. There exists

$$0 < \delta < \inf_{x,y\in B\,,\,x\neq y} d(x,y) .$$

Let $k \in \mathbb{N}$ such that $\operatorname{diam}(B) \leqslant \delta$ for all $B \in P_\ell$ and $\ell \geqslant k$. For all $\ell \geqslant k$ and all $x \in B$, there exist $x \in B_x \in P_\ell$. Then $B_x \cap B_y = \varnothing$ for all $x \neq y$, $x, y \in B$. Hence $g_\ell(y) \geqslant \#B$, for all $\ell \geqslant k$. Thus, $N(f \mid A, \sqcup) = \sup_j g_j$ is $\mathcal{H}^s$ measurable. By corollary 1.5.6,

$$\int N(f \mid A, y)\, d\mathcal{H}^s(y) = \sup_j \sum_{B\in P_j} \mathcal{H}^s(f(B)) = \sup_j \sum_{B\in P_j} \sigma(B) .$$

Clearly, $\mu_\delta(A) \leqslant \sum_{B\in P_\ell} \sigma(B) \leqslant \sum_{B\in P_\ell} \mu(B) = \mu(A)$ for all $\ell \geqslant k$ (σ is σ-subadditive, and hence $\leqslant \mu$). Hence, the right hand side equals $\mu(A)$. Moreover,

$$\sup_j \sum_{B\in P_j} \sigma(B) \leqslant \operatorname{Lip}(f)^s \cdot \sup_j \sum_{P\in P_j} \mathcal{H}^s(B) = \operatorname{Lip}(f)^s \cdot \mathcal{H}^s(A) ,$$

hence the assertion. □

Corollary 2.2.3. Let X be separable and $f : X \to Y$ be Lipschitz, $A \subset X$ Borel and $\mathcal{H}^s(A) < \infty$. Then for $\mathcal{H}^s$ a.e. $y \in Y$, $A \cap f^{-1}(y)$ is countable.

Proof. By theorem 2.2.2, $N(f \mid A, \sqcup)$ is $\mathcal{H}^s$ summable, and thus $\mathcal{H}^s$ a.e. finite. □

Theorem 2.2.4. Let $0 \leqslant s < \infty$.

(i). $\mathcal{H}^0$ is counting measure.

(ii). $\mathcal{H}^1 = \mathcal{L}^1$ on $X = \mathbb{R}$.

(iii). If X is a separable normed vector space, then for $\lambda > 0$, $\mathcal{H}^s(\lambda \cdot A) = \lambda^s \cdot \mathcal{H}^s(A)$, and for $x \in X$, $\mathcal{H}^s(A + x) = \mathcal{H}^s(A)$.

(iv). On $\mathbb{R}^n$, whenever $s > n$, then $\mathcal{H}^s = 0$, and $\mathcal{H}^n$ is locally finite.

Proof of (i). Observe $\omega_0 = 1$, in fact $\varrho_0(A) = 1$ for all $A \neq \varnothing$. Since the set of all singletons in $A \subset X$ is a partition of A, we find for all countable A that

$$\mathcal{H}^0_\delta(A) \leqslant \#A = \sum_{a\in A} \varrho_0(a) \leqslant \sum_{a\in A} \mathcal{H}^0(a) = \mathcal{H}^0(A) .$$

If A is infinite, we thus have $\mathcal{H}^0(A) = \infty$, and if A is finite, $\mathcal{H}^0(A) = \#A$.

Proof of (ii). Note $\omega_1 = 2$, and let $A \subset \mathbb{R}$. By definition, $\mathcal{L}^1(A) \leqslant \mathcal{H}^1_\delta(A) \leqslant \mathcal{H}^1(A)$ for all $\delta > 0$. Fix $\delta > 0$, and a countable cover (A_k) of A, and let $A_{k\ell} = A_k \cap [\delta\ell, \delta(\ell+1)]$ for all $\ell \in \mathbb{Z}$. Then $(A_{k\ell})$ covers A, $\operatorname{diam} A_{k\ell} \leqslant \delta$, and $\sum_{\ell\in\mathbb{Z}} \operatorname{diam}(A_{k\ell}) \leqslant \operatorname{diam}(A_k)$ for all $k \in \mathbb{N}$. Thus,

$$\sum_{k=0}^{\infty} \operatorname{diam}(A_k) \geqslant \textstyle\sum_{(k,\ell)\in\mathbb{N}\times\mathbb{Z}} \operatorname{diam}(A_{k\ell}) \geqslant \mathcal{H}^1_\delta(A) .$$

Taking the infimum over all (A_k), we deduce $\mathcal{L}^1(A) \geqslant \mathcal{H}^1_\delta(A)$. Taking the supremum over δ, it follows that $\mathcal{H}^1(A) = \mathcal{L}^1(A)$.

Proof of (iii). This follows from theorem 2.2.2 by considering

$$d^\pm(y) = \lambda^{\pm 1} \cdot y \quad \text{and} \quad t^\pm(y) = x + y .$$

Proof of (iv). For fixed $m \geqslant 1$, $C = [0,1]^n$ is the union of the $C(a,b) = \prod_{j=1}^n [a_j, b_j]$ such that $0 \leqslant a_j \leqslant b_j \leqslant m$ and $m \cdot (b_j - a_j) = 1$. Then $\operatorname{diam} C(a,b) = \frac{\sqrt{n}}{m}$, and the number of such cubes is m^n. Hence,

$$\mathcal{H}^s_{\sqrt{n}/m}(C) \leqslant \frac{\omega_s \cdot n^{s/2} \cdot m^{n-s}}{2^s} .$$

It follows that $\mathcal{H}^s(C) = 0$ for $s > n$, and $\mathcal{H}^n(C) \leqslant \frac{\omega_s \cdot n^{s/2}}{2^s} < \infty$. Since $\mathbb{R}^n$ is the countable union of copies of C, and translates of scaled versions of C exhaust a neighbourhood basis, this entails the claim. □

We note the following lemma.

Lemma 2.2.5. If $A \subset X$ and $\mathcal{H}^s_\delta(A) = 0$ for some δ, then $\mathcal{H}^s(A) = 0$.

Proof. Let $\varepsilon > 0$, w.l.o.g. $s > 0$. Then there exists a countable cover $\mathcal{G}$ of A such that

$$\textstyle\sum_{G\in\mathcal{G}} \frac{\omega_s}{2^s} \cdot \operatorname{diam}(G)^s \leqslant \varepsilon .$$

Then $\operatorname{diam}(G) \leqslant \delta_\varepsilon = 2\sqrt[s]{\frac{\varepsilon}{\omega_s}} \to 0$, so we find $\mathcal{H}^s(A) = \lim_{\varepsilon\to 0+} \mathcal{H}^s_{\delta_\varepsilon}(A) = 0$. □

Proposition 2.2.6. Let $A \subset X$ and $0 \leqslant s < t < \infty$. Then $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$.

Proof. Given $\delta > 0$, choose a countable cover $\mathcal{G}$ of A, $\operatorname{diam}(G) \leqslant \delta$ for all $G \in \mathcal{G}$, with

$$\textstyle\sum_{G\in\mathcal{G}} \frac{\omega_s}{2^s} \cdot \operatorname{diam}(G)^s \leqslant \mathcal{H}^s_\delta(A) + 1 \leqslant \mathcal{H}^s(A) + 1 .$$

Then

$$\mathcal{H}^t_\delta(A) \leqslant \textstyle\sum_{G\in\mathcal{G}} \frac{\omega_t}{2^t} \cdot \operatorname{diam}(G)^t \leqslant \frac{\omega_t \cdot \delta^{t-s}}{\omega_s \cdot 2^{t-s}} \cdot \big(\mathcal{H}^s(A) + 1\big) .$$

Hence, $\mathcal{H}^s(A) = 0$, as asserted. □

The previous results motivate the following definition.

Definition 2.2.7. If X is metric, then its *Hausdorff dimension* is defined to be

$$\dim_{\mathcal{H}} X = \inf\{s \in [0, \infty[\mid \mathcal{H}^s(X) = 0\} \in [0, \infty] .$$

Clearly, $\dim_{\mathcal{H}} Y \leqslant \dim_{\mathcal{H}} X$ for all $Y \subset X$. Also $\dim_{\mathcal{H}} \mathbb{R}^n \leqslant n$ by theorem 2.2.4 (iv). But it is easy to see that $\mathcal{H}^n(\mathbb{R}^n) = \infty$, so $\dim_{\mathcal{H}} \mathbb{R}^n = n$. (In fact, we shall see below that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.)

We digress briefly to construct 'fractal' sets of non-integral Hausdorff dimension.

2.2.8. Fix $0 < \lambda < \frac{1}{2}$. Inductively, define for all $k \in \mathbb{N}$ intervals I_{kj}, $1 \leqslant j \leqslant 2^k$, as follows. Let $I_{0,1} = [0,1]$. Then, for $k \geqslant 2$, $1 \leqslant j \leqslant 2^{k-1}$, let $I_{k,2j-1}$, $I_{k,2j}$, be the closed intervals of diameter λ^k obtained from $I_{k-1,j}$ by deleting an open interval of length $(1-2\lambda) \cdot \lambda^{k-1}$ at the centre of $I_{k-1,j}$, such that $\max I_{k,2j-1} < \min I_{k,2j}$. The compact

$$C_\lambda = \bigcap_{k=0}^{\infty} \bigcup_{j=1}^{2^k} I_{k,j} \subset [0,1]$$

is called the *Cantor λ-set*.

Proposition 2.2.9. Let $0 < \lambda < \frac{1}{2}$. Then C_λ has the Hausdorff dimension $s = -\frac{\log 2}{\log \lambda}$. In particular, any number in $[0,1]$ is attained as the Hausdorff dimension of a subset of $\mathbb{R}$.

Proof. Let $t \in \mathbb{R}$. Observe $\operatorname{diam}(I_{k,j}) = \lambda^k$, so

$$\mathcal{H}^t_{\lambda^k}(C_\lambda) \leqslant \omega_t \sum_{j=1}^{2^k} \frac{\lambda^{tk}}{2^t} = \frac{\omega_t}{2^t} \cdot (2\lambda^t)^k .$$

Whenever $t > s$, then $2\lambda^t < 1$, so $\mathcal{H}^t(C_\lambda) = 0$. Moreover, $\mathcal{H}^s(C_\lambda) \leqslant \frac{\omega_s}{2^s}$. To complete our proof, we claim that $\mathcal{H}^s(C_\lambda) \geqslant \frac{\omega_s}{2^{s+2}} > 0$.

Let $\mathcal{I}$ be a countable cover of C_λ by open intervals. Since C_λ is compact, we may assume $\mathcal{I}$ is finite. Let $\varepsilon > 0$. The set C_λ is meager, so by enlarging the diameter of each $I \in \mathcal{I}$ by at most $\sqrt[s]{\operatorname{diam}(I) + \frac{2^s\varepsilon}{\omega_s \# \mathcal{I}}} - \operatorname{diam}(I)$, we obtain a new covering $\mathcal{I}^\varepsilon$ such that the end points do not belong to C_λ, and

$$\textstyle\sum_{I \in \mathcal{I}} \varrho_s(I) + \varepsilon \geqslant \sum_{I \in \mathcal{I}^\varepsilon} \varrho_s(I) .$$

Hence, for some $\delta > 0$, the end points of all $I \in \mathcal{I}^\varepsilon$ are at least at a distance of δ from the set C_λ. There exists $n \in \mathbb{N}$, such that $\lambda^k < \delta$ for all $k \geqslant n$. Then any $I_{k,j}$ with $k \geqslant n$ is contained in some $I \in \mathcal{I}^\varepsilon$. Let $I \in \mathcal{I}^\varepsilon$ and fix $\ell \geqslant n$ such that $I_{\ell,p} \subset I$ for some p. Let $P = \{1 \leqslant p \leqslant 2^\ell | I_{\ell,p} \subset I\}$. Now let k be minimal, such that there is some q for which $I_{k,q} \subset I$. Set $Q = \{1 \leqslant q \leqslant 2^k | I_{k,q} \subset I\}$.

We claim that $\#Q \leqslant 2$. Indeed, any interval containing intervals $I_{k,i}$ and $I_{k,j}$ contains all $I_{k,m}$, $i \leqslant m \leqslant j$. Moreover, an interval containing $I_{k,2j-1}$ and $I_{k,2j}$ also contains $I_{k-1,j}$. Thus, if I would contain at least three $I_{k,j}$, then it would contain some $I_{k-1,i}$, contrary to the minimality of k. Moreover, for all $p \in P$, there exists some q such that $q-1$, q or $q+1$ belongs to Q and $I_{\ell,p} \subset I_{k,q}$ (by the same argument).

Hence,

$$2\varrho_s(I) \geqslant \textstyle\sum_{q\in Q} \varrho_s(I_{k,j}) \geqslant \sum_{p\in P} \varrho_s(I_{\ell,i}) - 2\varrho_s(I_{k,1}) \geqslant \sum_{p\in P} \varrho_s(I_{\ell,i}) - 2\varrho_s(I) .$$

This implies

$$\textstyle\sum_{I\in\mathcal{I}} \varrho_s(I) + \varepsilon \geqslant \sum_{I\in\mathcal{I}^\varepsilon} \varrho_s(I) \geqslant \dfrac{1}{4} \cdot \sum_{I\in\mathcal{I}^\varepsilon} \sum_{I_{m,p}\subset I} \varrho_s(I_{m,p}) \geqslant \dfrac{1}{4} \displaystyle\sum_{p=1}^{2^\ell} \varrho_s(I_{\ell,p}) = \frac{\omega_s}{2^{s+2}} .$$

Since $\varepsilon > 0$ and $\mathcal{I}$ were arbitrary, we find $\mathcal{H}^s(C_\lambda) \geqslant \frac{\omega_s}{2^{s+2}}$ as claimed. □

Remark 2.2.10. The above theorems are a mixture of [Fed69] and [EG92]. The proof of proposition 2.2.9 follows [Mat95, 4.11], although the proof given there is erroneous.

2.3 Densities

Definition 2.3.1. Let μ be a measure on X and $0 \leqslant s < \infty$. Define the *upper and lower s-density of μ at $x \in X$* by

$$\Theta^*_s(\mu, x) = \limsup_{\varepsilon\to 0+} \frac{\mu\big(B(x,\varepsilon)\big)}{\omega_s \varepsilon^s} \quad \text{and} \quad \Theta_{*s}(\mu, x) = \liminf_{\varepsilon\to 0+} \frac{\mu\big(B(x,\varepsilon)\big)}{\omega_s \varepsilon^s} .$$

Whenever these numbers agree in $[0,\infty]$, define the *s-dimensional density of μ at x*, denoted $\Theta_s(\mu, x)$, to be their common value. If $\mu = \mathcal{H}^s \llcorner A$, we write $\Theta^*_s(A, x)$, etc.

2.3.2. Before stating our main theorem on densities, we make the following simple observation: A point $x \in X$ is not isolated if and only if there exists a sequence $r_k \to 0+$ such that $\operatorname{diam}\big(B(x, r_k)\big) \geqslant r_k$ for all $k \in \mathbb{N}$.

Indeed, if $x \in X$ is isolated, then there exists $R > 0$ such that $B(x, R) = x$. Then $B(x, r) = x$ for all $0 < r \leqslant R$, so there exists no such sequence. Conversely, assume that $x \in X$ is not isolated. Then there are $x_k \in B\big(x, \frac{1}{k}\big) \setminus x$. Let $r_k = d(x, x_k)$. Then r_k has the required properties.

Theorem 2.3.3. Let μ be a measure, $B \subset X$, and $0 < \alpha < \infty$. Then

$$\Theta^*_s(\mu, x) \leqslant \alpha \text{ for all } x \in B \quad \Rightarrow \quad \mu \llcorner B \leqslant 2^s \alpha \cdot \mathcal{H}^s \llcorner B .$$

If, moreover, X is separable, μ is finite and Borel regular, and B does not contain an isolated point of X, then

$$\Theta^*_s(\mu, x) \geqslant \alpha \text{ for all } x \in B \quad \Rightarrow \quad \mu \llcorner B \geqslant \alpha \cdot \mathcal{H}^s \llcorner B .$$

Remark 2.3.4. The point of the theorem is that we do not need any additional assumptions on the separable metric space X, whereas in lemma 1.9.2, we needed σ-directional limitation.

Proof of theorem 2.3.3. Let $\Theta^*_s(\mu, \llcorner\!\lrcorner) \leqslant \alpha$ on B. Let $A \subset X$. There exists a countable cover $\mathcal{G}$ of $A \cap B$ such that

$$\forall\, G \in \mathcal{G}\, \exists\, x_G \in G \cap A \cap B\,,\ 0 < r_G = \mathrm{diam}(G) < \infty\,,\ \text{and}\ \mathcal{H}^s(A \cap B) + \varepsilon \geqslant \textstyle\sum_{G \in \mathcal{G}} \varrho_s(G)\,.$$

Then

$$\begin{aligned}(\mu \llcorner B)(A) &\leqslant \textstyle\sum_{G\in\mathcal{G}} (\mu \llcorner B)(A \cap G) \leqslant \sum_{G\in\mathcal{G}} \mu\big(B(x_G, r_G)\big) \\ &\leqslant \alpha \cdot \textstyle\sum_{G\in\mathcal{G}} \omega_s r_G^s = 2^s\alpha \cdot \sum_{G\in\mathcal{G}} \varrho_s(G) \leqslant 2^s\alpha \cdot \big(\mathcal{H}^s(A\cap B) + \varepsilon\big)\,,\end{aligned}$$

proving the first assertion.

Let $\Theta^*_s(\mu, \llcorner\!\lrcorner) \geqslant \alpha$ on B, and make the additional assumptions. Choose $1 < \tau < \infty$ and let $\eta = 2(2 + 2\tau)$. Then, for $x \in B$, and arbitrarily small $r > 0$,

$$\mathrm{diam}\big(B(x, (1+2\tau)r)\big) \leqslant 2(1+2\tau)r < \eta \cdot r \leqslant \eta \cdot \mathrm{diam}\big(B(x,r)\big)\,.$$

Let $A \subset X$. We may assume that A is Borel, because μ and $\mathcal{H}^s \llcorner B$ are Borel regular. By theorem 1.1.9 (i), there exists an open $U \supset A \cap B$ such that $\mu(U) \leqslant \mu(A \cap B) + \frac{\varepsilon}{2}$. Fix $\delta > 0$, and let $\mathcal{F}$ be set of all $B(x,r)$, $x \in A \cap B$, $r \leqslant \frac{\delta}{\eta}$, subject to the conditions

$$B\big(x, (1+2\tau)r\big) \subset U\,,\ \mathrm{diam}\big(B(x,r)\big) \geqslant r\,,\ \text{and}\ \mu\big(B(x,r)\big) \geqslant \alpha \cdot \omega_s \cdot r^s > 0\,.$$

By assumption, $\mathcal{F}$ is a closed fine cover of $A \cap B$, and

$$\delta(\hat{F}) < \eta \cdot \delta(F) \quad \text{for all } F \in \mathcal{F} \quad \text{where} \quad \delta(F) = \mathrm{diam}(F)\,.$$

Thus, by corollary 1.7.8, there exists a disjoint $\mathcal{G} \subset \mathcal{F}$, such that

$$(A \cap B) \setminus \bigcup \mathcal{H} \subset \bigcup \{\hat{G} \mid G \in \mathcal{G} \setminus \mathcal{H}\} \quad \text{for all finite} \quad \mathcal{H} \subset \mathcal{G}\,.$$

We have $\mu(U) < \infty$ and $\mu(G) > 0$ for all $G \in \mathcal{G}$. Hence, $\mathcal{G}$ is countable, and we have $\sum_{G\in\mathcal{G}} \mu(G) \leqslant \mu(U) < \infty$. Choose $\mathcal{H} \subset \mathcal{G}$ finite such that $\sum_{G\in\mathcal{G}\setminus\mathcal{H}} \mu(G) \leqslant \frac{\varepsilon}{2\eta^s}$. Then

$$\begin{aligned}\alpha \cdot \mathcal{H}^s_\delta(A \cap B) &\leqslant \textstyle\sum_{G\in\mathcal{H}} \alpha\varrho_s(G) + \sum_{G\in\mathcal{G}\setminus\mathcal{H}} \alpha\varrho_s(\hat{G}) \\ &\leqslant \frac{1}{2^s} \textstyle\sum_{G\in\mathcal{H}} \mu(G) + \sum_{G\in\mathcal{G}\setminus\mathcal{H}} \eta^s \alpha\mu(G) \leqslant \mu(U) + \dfrac{\varepsilon}{2} \leqslant \mu(A\cap B) + \varepsilon\,,\end{aligned}$$

proving the assertion. ☐

Corollary 2.3.5. Assume that X is separable, and contains no isolated points. If μ is Borel regular, and $A \in \mathfrak{M}(\mu)$, $\mu(A) < \infty$, then

$$\Theta_s^*(\mu \llcorner A, x) = 0 \quad \text{for } \mathcal{H}^s \text{ a.e. } x \in X \setminus A \,.$$

Proof. By proposition 1.1.7, $\mu \llcorner A$ is a finite Borel measure. For any $n \geqslant 1$, let

$$B_n = (X \setminus A) \cap \left\{ \Theta_s^*(\mu, \sqcup) > \frac{1}{n} \right\} .$$

Assume $\mathcal{H}^s(B_n) > 0$. Since $\mu(A) < \infty$, there exists by lemma 1.1.8 a closed $C \supset B_n$ such that $\mu(A \setminus C) < \frac{1}{n} \cdot \mathcal{H}^s(B_n)$. But theorem 2.3.3 gives

$$\mu(A \setminus C) = (\mu \llcorner A)(X \setminus C) \geqslant \frac{1}{n} \cdot \mathcal{H}^s(X \setminus C) \geqslant \frac{1}{n} \cdot \mathcal{H}^s(B_n) \,,$$

a contradiction! Thus $\mathcal{H}^s(B_n) = 0$, and taking unions, the claim follows. □

Corollary 2.3.6. Assume that X is separable, and contains no isolated points. Whenever $\mathcal{H}^s(A) < \infty$, then

$$\Theta_s^*(A, x) \leqslant 1 \quad \text{for } \mathcal{H}^s \text{ a.e. } x \in X \,.$$

Proof. Since $\mathcal{H}^s$ is Borel regular, we may assume that A is Borel. For $n \geqslant 1$, define $B_n = \left\{ \Theta_s^*(A, \sqcup) > 1 + \frac{1}{n} \right\}$. Since $\mathcal{H}^s \llcorner A$ is Borel regular and finite, theorem 2.3.3 gives

$$\infty > \mathcal{H}^s(B_n) \geqslant (\mathcal{H}^s \llcorner A)(B_n) \geqslant \left(1 + \frac{1}{n}\right) \cdot \mathcal{H}^s(B_n) \,,$$

so $\mathcal{H}^s(B_n) = 0$. Taking intersections, the assertion follows. □

Remark 2.3.7. The proof of the density results is from [Fed69]. The proof in [Mat95, th. 6.2.] is erroneous; it assumes that $\mathcal{H}^s$ be a Radon measure, which is usually false.

2.4 Isodiametric Inequality and Uniqueness of Measure on $\mathbb{R}^n$

It is an important and striking fact that for any norm $\|\sqcup\|$ on $\mathbb{R}^n$, the Hausdorff measure $\mathcal{H}^n_{\|\sqcup\|}$ equals the Euclidean Lebesgue measure $\mathcal{L}^n$ up to a fixed constant. The main ingredient in its proof, the isodiametric inequality, states that the $\|\sqcup\|$-unit ball is the set of maximal Lebesgue measure among the sets of $\|\sqcup\|$-diameter 2 . To prove this, we first establish the Brunn-Minkowski inequality.

Theorem (Brunn-Minkowski-Lusternik) 2.4.1. Let $\varnothing \neq A, B \subset \mathbb{R}^n$ be $\mathcal{L}^n$ measurable, and $t \in [0,1]$. If $tA + (1-t)B$ is $\mathcal{L}^n$ measurable, then

$$\mathcal{L}^n\big(tA + (1-t)B\big)^{1/n} \geqslant t\mathcal{L}^n(A)^{1/n} + (1-t)\mathcal{L}^n(B)^{1/n} \,.$$

Proof. We shall assume, as we may w.l.o.g., that $0 < t < 1$.

First, we shall prove the statement for boxes whose sides are parallel to the coordinate hyperplanes. By Fubini's theorem 1.6.2, for any box A of side lengths $0 < x_j < \infty$, we have $\mathcal{L}^n(B) = \prod_{j=1}^n x_j$. If $A = a + \prod_{j=1}^n [0, x_j]$ and $B = b + \prod_{j=1}^n [0, y_j]$, then $A + B = a + b + \prod_{j=1}^n [0, x_j + y_j]$, so $\mathcal{L}^n(A + B) = \prod_{j=1}^n (x_j + y_j)$.

Recall that the arithmetic-geometric mean inequality states

$$\sqrt[n]{a_1 \cdots a_n} \leqslant \frac{1}{n} \cdot \big(a_1 + \cdots + a_n\big) \quad \text{for all } a_j \geqslant 0 .$$

It gives

$$\begin{aligned} \frac{t\mathcal{L}^n(A)^{1/n} + (1-t)\mathcal{L}^n(B)^{1/n}}{\mathcal{L}^n\big(tA + (1-t)B\big)} &= \Big(\prod_{j=1}^n \frac{tx_j}{tx_j + (1-t)y_j}\Big)^{1/n} + \Big(\prod_{j=1}^n \frac{(1-t)y_j}{tx_j + (1-t)y_j}\Big)^{1/n} \\ &\leqslant \frac{1}{n} \cdot \sum_{j=1}^n \frac{tx_j}{tx_j + (1-t)y_j} + \frac{1}{n} \cdot \sum_{j=1}^n \frac{(1-t)y_j}{tx_j + (1-t)y_j} = 1 , \end{aligned}$$

so the statement is correct for boxes.

We now prove the statement for finite unions of boxes. So, let A and B be finite unions of boxes mutually not intersecting in their interiors, such that that the total number is at most $k > 2$, and assume the inequality has been established for unions whose total number of boxes is $< k$.

By possibly exchanging A and B , w.l.o.g. there exists an affine hyperplane H parallel to the coordinate hyperplanes, such that $A^{\pm} = H^{\pm} \cap A$ are unions of fewer boxes than A is, where $H^{\pm}$ are the closed half spaces defined by H . (Otherwise, $k \leqslant 2$, which we have excluded.) [1] For $B^{\pm} = G^{\pm} \cap B$, where G is some translate of H , we have

$$\frac{\mathcal{L}^n(A^{\pm})}{\mathcal{L}^n(A)} = \frac{\mathcal{L}^n(B^{\pm})}{\mathcal{L}^n(B)} .$$ [2]

[1] The proof of the existence of the hyperplane H goes as follows. Observe that if $P = a + \prod_{j=1}^n [0, x_j]$ and $Q = b + \prod_{j=1}^n [0, y_j]$ are two boxes, then for Q to not to be on the opposite side as P of the hyperplane $\{\mathrm{pr}_j = a_j\}$, say, amounts to the inequality $x_j \geqslant a_j$. For Q not to be on the opposite side of each of the $2n$ hyperplanes spanned by the top-dimensional faces of P amounts to $2n$ inequalities, which then gives an n-dimensional intersection $P \cap Q$. If P and Q only intersect in their boundaries if they are distinct, we find $P = Q$. Thus, w.l.o.g., let A be the union of at least two boxes. Then there exists an affine hyperplane H parallel to one of the coordinate hyperplanes, such that at least two of the boxes are completely on opposite sides of H . Every other box is either dissected by H , in which case it contributes to both A^+ and A^- , or lies on one side of H , in which case it contributes only to A^+ or only to A^- . Since at least one box only contributes to either side, the total number of boxes in $A^{\pm}$ is strictly less than in A .

[2] Say $H = \{pr_1 = 0\}$. Consider $B_t^+ = B \cap \{\mathrm{pr}_1 \leqslant t\}$. Then

$$f(t) = \mathcal{L}^n(B_t^+) = \int_{-\infty}^t \mathcal{L}^{n-1}(B \cap \{\mathrm{pr}_1 = t\})\, d\mathcal{L}^1(t)$$

(Fubini) is continuous by Lebesgue's theorem. The intermediate value theorem gives one equation, and the other follows by additivity of $\mathcal{L}^n$, because $A^+ \cap A^-$ and $B^+ \cap B^-$ are $\mathcal{L}^n$ negligible.

Then $B^{\pm}$ are unions of at most as many boxes as B is. Hence, $A^{\varepsilon} \cup B^{\varepsilon}$ are unions of $< k$ boxes for $\varepsilon^2 = 1$. This implies

$$\begin{aligned}
\mathcal{L}^n(tA + (1-t)B) &\geqslant \sum_{\varepsilon^2=1} \mathcal{L}^n(tA^{\varepsilon} + (1-t)B^{\varepsilon}) \geqslant \sum_{\varepsilon^2=1} \big(t\mathcal{L}^n(A^{\varepsilon})^{1/n} + (1-t)\mathcal{L}^n(B^{\varepsilon})^{1/n}\big)^n \\
&= \Big(1 + \frac{(1-t)\mathcal{L}^n(B)^{1/n}}{t\mathcal{L}^n(A)^{1/n}}\Big)^n \cdot \sum_{\varepsilon^2=1} t^n \mathcal{L}^n(A^{\varepsilon}) \\
&= \Big(1 + \frac{(1-t)\mathcal{L}^n(B)^{1/n}}{t\mathcal{L}^n(A)^{1/n}}\Big)^n \cdot t^n \mathcal{L}^n(A) = \big(t\mathcal{L}^n(A)^{1/n} + (1-t)\mathcal{L}^n(B)^{1/n}\big)^n ,
\end{aligned}$$

which proves the assertion for finite unions of boxes.

Next, assume that A, B be compact. Let $\mathcal{G}_j$ and $\mathcal{H}_j$ be families of boxes mutually intersecting only along their boundaries, such that

$$A \subset A_{j+1} \subset A_j = \bigcup \mathcal{G}_j \quad \text{and} \quad B \subset B_{j+1} \subset B_j = \bigcup \mathcal{H}_j .$$

Then

$$\begin{aligned}
t\mathcal{L}^n(A)^{1/n} + (1-t)\mathcal{L}^n(B)^{1/n} &\leqslant \limsup_j \big(t\mathcal{L}^n(A_j)^{1/n} + (1-t)\mathcal{L}^n(B_j)^{1/n}\big) \\
&\leqslant \limsup_j \mathcal{L}^n\big(tA_j + (1-t)B_j\big)^{1/n} .
\end{aligned}$$

We may assume that $A_j \subset A^{1/2j}$ and $B_j \subset B^{1/2j}$, where $C^{\varepsilon} = \{c \in \mathbb{R}^n | \operatorname{dist}(c, C) \leqslant \varepsilon\}$. If $\mathcal{G}$ is any cover of C, then $\{G^{\varepsilon} \mid G \in \mathcal{G}\}$ is a cover of C, so $\mathcal{L}^n(C) \leqslant (1+2\varepsilon)^n \mathcal{L}^n(C)$. We find $tA_j + (1-t)B_j \subset \big(tA + (1-t)B\big)^{1/2j}$, and hence

$$\limsup_j \mathcal{L}^n\big(tA_j + (1-t)B_j\big) \leqslant \limsup_j \big(1 + \tfrac{1}{j}\big)^n \cdot \mathcal{L}^n(tA + (1-t)B) ,$$

which proves the assertion for compact A, B.

Finally, let A, B, and $tA + (1-t)B$ be $\mathcal{L}^n$ measurable. Of course, we may assume $\mathcal{L}^n(tA + (1-t)B) < \infty$. By theorem 1.1.9 (iii), let $A \setminus \bigcup_{j=0}^{\infty} A_j$ and $B \setminus \bigcup_{j=0}^{\infty} B_j$ be $\mathcal{L}^n$ negligible, where $A_j \subset A_{j+1}$ and $B_j \subset B_{j+1}$ are compacts.

Then $C_j = tA_j + (1-t)B_j \subset C_{j+1}$, and $(tA + (1-t)B) \setminus \bigcup_{j=0}^{\infty} C_j$ is $\mathcal{L}^n$ negligible. By proposition 1.1.3, we have

$$\begin{aligned}
\mathcal{L}^n\big(tA + (1-t)B\big)^{1/n} &= \lim_k \mathcal{L}^n(C_j) \geqslant \lim_k t\mathcal{L}^n(A_j)^{1/n} + \lim_j (1-t)\mathcal{L}^n(B_j)^{1/n} \\
&= t\mathcal{L}^n(A)^{1/n} + (1-t)\mathcal{L}^n(B)^{1/n} ,
\end{aligned}$$

finally proving the theorem. □

Isodiametric Inequality (Bieberbach-Urysohn-Mel'nikov) 2.4.2. Let $\|\sqcup\|$ be any norm on $\mathbb{R}^n$, $A \subset \mathbb{R}^n$, and $2r = \operatorname{diam}(A)$, taken w.r.t. $\|\sqcup\|$. Then

$$\mathcal{L}^n(A) \leqslant \mathcal{L}^n\{\|\sqcup\| \leqslant r\} .$$

Proof. We may assume $r < \infty$ and A closed ($\mathcal{L}^n(A) \leqslant \mathcal{L}^n(\overline{A})$ and $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$). So A is compact, and $B = \frac{1}{2}(A - A)$ gives $\operatorname{diam}(B) \leqslant \operatorname{diam}(A)$. Moreover, theorem 2.4.1 shows

$$\mathcal{L}^n(B) \geqslant \left(\tfrac{1}{2} \cdot \mathcal{L}^n(A)^{1/n} + \tfrac{1}{2} \cdot \mathcal{L}^n(-A)^{1/n}\right)^n = \mathcal{L}^n(A) .$$

But since $B = -B$, we find for any $x \in B$ that $2\|x\| = \|x - (-x)\| \leqslant \operatorname{diam} B \leqslant 2r$, so B is contained the ball of radius r . We conclude

$$\mathcal{L}^n(A) \leqslant \mathcal{L}^n(B) \leqslant \mathcal{L}^n\{\|\sqcup\| \leqslant r\} ,$$

which was our claim. □

Theorem 2.4.3. Let $\|\sqcup\|$ be a norm on $\mathbb{R}^n$ and $\mathcal{H}^n = \mathcal{H}^n_{\|\sqcup\|}$ the n-dimensional Hausdorff measure w.r.t. this norm. Then $\mathcal{H}^n = \frac{\omega_n}{\mathcal{L}^n\{\|\sqcup\|\leqslant 1\}} \cdot \mathcal{L}^n$, in particular, $\mathcal{H}^n\{\|\sqcup\| \leqslant 1\} = \omega_n$ is independent of the norm.

Proof. Let $B(x,r) = \{\|\sqcup - x\| \leqslant r\}$ be the ball w.r.t. $\|\sqcup\|$.

Theorem 2.2.2 implies that for all $x \in \mathbb{R}^n$, $\lambda > 0$, $\mathcal{H}^n(B(x,r)) = \lambda^n \cdot \mathcal{H}^n(B(0,1))$. By theorem 2.2.4 (iv), $\mathcal{H}^n$ is locally finite. Thus, for $C = \frac{\mathcal{H}^n(B(0,1))}{\mathcal{L}^n(B(0,1))}$, we have $\mathcal{H}^n = C \cdot \mathcal{L}^n$, by corollary 1.9.6.

The isodiametric inequality (theorem 2.4.2) implies that for any countable cover $\mathcal{G}$ of $B(0,1)$ such that $\operatorname{diam}(G) \leqslant \delta$ for all $G \in \mathcal{G}$,

$$\mathcal{L}^n\big(B(0,1)\big) \leqslant \textstyle\sum_{G\in\mathcal{G}} \mathcal{L}^n(G) \leqslant \dfrac{\mathcal{L}^n(0,1)}{\omega_n} \cdot \textstyle\sum_{G\in\mathcal{G}} \dfrac{\operatorname{diam}(G)^n \cdot \omega_n}{2^n} ,$$

so $\omega_n \leqslant \mathcal{H}^n_\delta\big(B(0,1)\big) \leqslant \mathcal{H}^n\big(B(0,1)\big)$. In particular, $\mathcal{H}^n \neq 0$.

Then corollary 2.3.6 gives

$$\mathcal{H}^n\big(B(x,1)\big) = \omega_n \cdot \limsup_{\varepsilon\to 0+} \frac{\mu\big(B(x,\varepsilon)\big)}{\omega_n \cdot \varepsilon^n} \leqslant \omega_n \quad \text{for } \mathcal{H}^n \text{ a.e. } x \in \mathbb{R}^n .$$

But the left hand side is independent of x , so $\mathcal{H}^n\big(B(0,1)\big) \leqslant \omega_n$, because $\mathcal{H}^n \neq 0$. Thus, $\mathcal{H}^n\big(B(0,1)\big) = \omega_n$, proving the theorem. □

Remark 2.4.4. Using the area formula (see section 4), one can prove that ω_n is the volume of the Euclidean unit ball, but we shall not use this result at this point.

The isodiametric inequality also implies the following generalisation of theorem 2.2.2.

Theorem 2.4.5. Let $m \leqslant s \leqslant n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz w.r.t. some arbitrary norms.

Then

$$\int \mathcal{H}^{s-m}\big(A \cap f^{-1}(y)\big)\, d\mathcal{L}^m(y) \leqslant \frac{\mathrm{Lip}(f)^m \cdot \mathcal{L}^m\{\|\sqcup\| \leqslant 1\} \cdot \omega_{s-m}}{\omega_s} \cdot \mathcal{H}^s(A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$ and the Hausdorff measures $\mathcal{H}^s$, $\mathcal{H}^{s-m}$ associated to the given norm.

Proof. For any $k \geqslant 1$, there exist countable covers $\mathcal{G}_k$ of A, such that

$$\textstyle\sum_{G \in \mathcal{G}_k} \varrho_s(G) \leqslant \mathcal{H}^s(A) + \frac{1}{k}\,.$$

If $y_1, y_2 \in f(G)$, $G \in \mathcal{G}_k$, then there exist $x_j \in G$, $f(x_j) = y_j$, $j = 1, 2$. Thus,

$$\|y_1 - y_2\| \leqslant \mathrm{Lip}(f) \cdot \|x_1 - x_2\| \leqslant \mathrm{Lip}(f) \cdot \operatorname{diam} G\,.$$

We obtain $\operatorname{diam} f(G) \leqslant \mathrm{Lip}(f) \cdot \operatorname{diam} G$, and thus

$$\mathcal{L}^m\big(f(G)\big) \leqslant \frac{L^m \cdot \alpha_m}{\omega_m} \cdot \varrho_m(G)\,, \quad \text{where} \quad L = \mathrm{Lip}(f) \quad \text{and} \quad \alpha_m = \mathcal{L}^m\{\|\sqcup\| \leqslant 1\}\,,$$

by the isodiametric inequality (theorem 2.4.2). Fatou's lemma (theorem 1.5.5) and monotone convergence (corollary 1.5.6) imply

$$\begin{aligned}
&\int \mathcal{H}^{s-m}\big(A \cap f^{-1}(y)\big)\, d\mathcal{L}^m(y) \\
&\quad\leqslant \liminf_k \int \mathcal{H}^{s-m}_{1/k}\big(A \cap f^{-1}(y)\big)\, d\mathcal{L}^m(y) \leqslant \liminf_k \int \textstyle\sum_{G \in \mathcal{G}_k} \varrho_{s-m}\big(G \cap f^{-1}(y)\big)\, \mathcal{L}^m(y) \\
&\quad\leqslant \liminf_k \textstyle\sum_{G \in \mathcal{G}_k} \varrho_{s-m}(G) \cdot \mathcal{L}^m\big(f(G)\big) \leqslant \dfrac{L^m \cdot \alpha_m}{\omega_m} \cdot \liminf_k \textstyle\sum_{G \in \mathcal{G}_k} (\varrho_{s-m} \cdot \varrho_m)(G) \\
&\quad\leqslant \dfrac{L^m \cdot \alpha_m \cdot \omega_{s-m}}{\omega_s} \cdot \liminf_k \textstyle\sum_{G \in \mathcal{G}_k} \varrho_s(G) \leqslant \dfrac{L^m \cdot \alpha_m \cdot \omega_{s-m}}{\omega_s} \cdot \mathcal{H}^s(A)\,,
\end{aligned}$$

proving the assertion. ☐

Corollary 2.4.6. Let $\|\sqcup\|$ be a seminorm on $\mathbb{R}^n$. There exists $L > 0$, such that for $\varepsilon \geqslant 0$,

$$\frac{2L}{\omega_n} \cdot \mathcal{H}^n\{\varepsilon \leqslant \|\sqcup\| \leqslant 1\} \geqslant \frac{1 - \varepsilon^n}{n \cdot \omega_{n-1}} \cdot \mathcal{H}^{n-1}\{\|\sqcup\| = 1\}\,,$$

for the Hausdorff measures $\mathcal{H}^n$, $\mathcal{H}^{n-1}$ to some norm.

Proof. Consider the L-Lipschitz map $\|\sqcup\| : \mathbb{R}^n \to \mathbb{R}$. Then $\mathcal{L}^1\big([-1, 1]\big) = 2$, and theorem 2.4.5 implies

$$\frac{2L \cdot \omega_{n-1}}{\omega_n} \cdot \mathcal{H}^n\{\|\sqcup\| \leqslant 1\} \geqslant \int_\varepsilon^1 \mathcal{H}^{n-1}\{\|\sqcup\| = r\}\, d\mathcal{L}^1(r) = \mathcal{H}^{n-1}\{\|\sqcup\| = 1\} \cdot \int_\varepsilon^1 r^{n-1}\, dr\,,$$

by theorem 2.2.4 (iii). Now, $\int_0^s r^{n-1}\, dr = \frac{s^n}{n}$ by standard arguments (Lebesgue's theorem and Lipschitz continuity of $\frac{1}{n}\,\mathrm{id}^n$). ☐

Remark 2.4.7. The constants in the above corollary can be removed to give an equality. This can be proved by using the area formula, but we shall use the corollary in the above form in the area formula's proof.

The proof of the Brunn-Minkowski and isodiametric inequalities is essentially from the book [BZ88]. The proof of theorem 2.4.3 is from [Kir94, lem. 6]. Theorem 2.4.5 is [Mat95, th. 7.7].

3 Lipschitz Extendibility and Differentiability

3.1 Extension of Lipschitz Functions

The most significant Lipschitz extendibility result is the Kirszbraun extension theorem on the extendibility of functions defined on subsets of finite dimensional Euclidean spaces. It was extended to infinite dimensions by Valentine. We give short proof based on the Fenchel duality theorem, as follows.

Theorem (Kirszbraun-Valentine) 3.1.1. Let $\mathcal{H}_j$, $j = 1, 2$, be Hilbert spaces, $\varnothing \neq D \subset \mathcal{H}_1$, and $f : D \to \mathcal{H}_2$ be Lipschitz. There exists a Lipschitz extension $g : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\mathrm{Lip}(g) = \mathrm{Lip}(f)$.

Proof. Assume that the theorem be true for $\mathcal{H}_1 = \mathcal{H}_2$. Then we may consider the function $h : D \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2 : (\xi, \eta) \mapsto (0, f(\xi))$ to extend it to the general situation. Moreover, it suffices to prove the assertion for $\mathrm{Lip}(f) = 1$. Indeed, the statement is trivial for $\mathrm{Lip}(f) = 0$, and we may consider $\mathrm{Lip}(f)^{-1} \cdot f$ for $\mathrm{Lip}(f) > 0$. □

The remainder of the proof is divided into a series of lemmata. To that end, let a subset $\varnothing \neq D \subset \mathcal{H} = \mathcal{H}_1 = \mathcal{H}_2$ and a 1-Lipschitz map $f : D \to \mathcal{H}$ be given, such that there exists no true 1-Lipschitz extension of f.

Moreover, define $\chi : \mathcal{H}^2 \to]-\infty, \infty]$ by

$$\chi(\xi, \eta) = \sup_{\zeta \in D} \big(\|\eta - f(\zeta)\|^2 - \|\xi - \zeta\|^2 \big) .$$

Lemma 3.1.2. We have $\chi \geqslant 0$ and $\{\chi = 0\} = \mathrm{Gr}(f)$, the graph of f.

Proof. Let $\xi, \eta \in \mathcal{H}$. If $\xi \in D$, then

$$\chi(\xi, \eta) \geqslant \|\eta - f(\xi)\|^2 - \|\xi - \xi\|^2 = \|\eta - f(\xi)\|^2 \geqslant 0 .$$

If $\xi \notin D$, then any extension of f to $D \cup \xi$ is not 1-Lipschitz. In particular, there exists $\zeta \in D$, such that $\|\eta - f(\zeta)\|^2 > \|\xi - \zeta\|^2$. Hence, $\chi(\xi, \eta) > 0$.

This proves the first assertion and that $\{\chi = 0\} \subset D \times \mathcal{H}$. Thus, $\chi(\xi, \eta) = 0$ implies, by the above calculation, that

$$0 = \chi(\xi, \eta) \geqslant \|\eta - f(\xi)\|^2 \geqslant 0 ,$$

so $(\xi, \eta) \in \mathrm{Gr}(f)$. Conversely, for $\eta = f(\xi)$, we have, for all $\zeta \in D$,

$$\|\eta - f(\zeta)\|^2 = \|f(\xi) - f(\zeta)\| \leqslant \|\xi - \zeta\|^2 ,$$

so $\chi(\xi, \eta) \leqslant 0$. Since $\chi(\xi, \eta)$, the assertion follows. □

Lemma 3.1.3. Let $\varphi : \mathcal{H}^2 \to]-\infty, \infty]$ be defined by

$$\varphi(\xi, \eta) = \frac{1}{4} \cdot \chi(\xi + \eta, \xi - \eta) + (\xi : \eta) \quad \text{for all } \xi, \eta \in \mathcal{H} .$$

Then the following holds.

(i). We have, for all $\xi, \eta \in \mathcal{H}$,

$$4 \cdot \varphi(\xi, \eta) = \sup_{\zeta \in D} \big(\|f(\zeta)\|^2 - \|\zeta\|^2 + 2 \cdot (\xi : \zeta - f(\zeta)) + 2 \cdot (\eta : \zeta + f(\zeta)) \big) .$$

(ii). For all $\zeta \in D$,

$$4 \cdot \varphi\Big(\frac{\zeta + f(\zeta)}{2}, \frac{\zeta - f(\zeta)}{2} \Big) = \|\zeta\|^2 - \|f(\zeta)\|^2 .$$

(iii). φ is a proper l.s.c. convex function, and for the Fenchel dual, we have $\varphi^* \geqslant \varphi$.

Proof of (i). For all ζ, we have

$$\begin{aligned} &\|\xi - \eta - f(\zeta)\|^2 - \|\xi + \eta - \zeta\|^2 \\ &\qquad = -4(\xi : \eta) - 2(\xi - \eta : f(\zeta)) + 2(\xi + \eta : \zeta) + \|f(\zeta)\|^2 - \|\zeta\|^2 \\ &\qquad = -4(\xi : \eta) + 2(\xi : \zeta - f(\zeta)) + 2(\eta : \zeta + f(\zeta)) + \|f(\zeta)\|^2 - \|\zeta\|^2 , \end{aligned}$$

and hence our claim follows.

Proof of (ii). We have

$$\begin{aligned} 4 \cdot \varphi\Big(\frac{\zeta + f(\zeta)}{2}, \frac{\zeta - f(\zeta)}{2} \Big) &= \chi\big(\zeta, f(\zeta)\big) + (\zeta + f(\zeta) : \zeta - f(\zeta)) \\ &= (\zeta + f(\zeta) : \zeta - f(\zeta)) = \|\zeta\|^2 - \|f(\zeta)\|^2 . \end{aligned}$$

Proof of (iii). The lower semicontinuity is clear from (1). The properness follows, since $\varphi > -\infty$ everywhere is obvious, and φ attains finite values by (2). Its convexity also follows from (1), since this exhibits φ as an upper envelope of affine functions.

Moreover, for all $\zeta \in D$,

$$\varphi^*(\eta, \xi) \geqslant \frac{1}{2} \cdot \big((\zeta + f(\zeta)) \oplus (\zeta - f(\zeta)) : \eta \oplus \xi \big) - \varphi\Big(\frac{\zeta + f(\zeta)}{2}, \frac{\zeta - f(\zeta)}{2} \Big) ,$$

and the supremum of the right hand side is $\varphi(\xi, \eta)$. □

Proof of theorem 3.1.1 (continued). Let $f : D \to \mathcal{H}$ be as before. We wish to show that $D = \mathcal{H}$. First, we establish $0 \in D$. To that end, we note that $h(\xi, \eta) = \frac{1}{2} \cdot \|\xi \oplus \eta\|^2$

equals its own Fenchel dual. Moreover,

$$\begin{aligned}4\cdot\big(\varphi(\xi,\eta)+h(\xi,\eta)\big)&=\chi(\xi+\eta,\xi-\eta)+4\cdot(\xi:\eta)+2\cdot\big(\|\xi\|^2+\|\eta\|^2\big)\\&=\chi(\xi+\eta,\xi-\eta)+2\cdot\|\xi+\eta\|^2\,.\end{aligned}\tag{$*$}$$

This implies

$$\varphi^*(\eta,\xi)+h(\eta,\xi)\geqslant\varphi(\xi,\eta)+h(\xi,\eta)\geqslant 0\,,$$

by lemma 3.1.3 (iii).

Then Fenchel's theorem gives $(\xi,\eta)\in\mathcal{H}\times\mathcal{H}$ such that $\varphi(\xi,\eta)+h(-\xi,-\eta)\leqslant 0$. From the equation ($*$), we find $\xi=-\eta$ and $\chi(0,\xi-\eta)=0$. This implies, by lemma 3.1.2, that $(0,\xi-\eta)\in\mathrm{Gr}(f)$, in particular, $0\in D$.

For any $\xi\in\mathcal{H}$, we may now consider the function $f_\xi:D-\xi\to\mathcal{H}$, defined by $f_\xi(\eta)=f(\eta+\xi)$. Then f_ξ satisfies all the conditions imposed on f, and we deduce $0\in D-\xi$, so $\xi\in D$. Thus, $D=\mathcal{H}$.

Let now $f:D\to\mathcal{H}$ be any 1-Lipschitz map, $D\neq\varnothing$. The set of all pairs (E,g) where $D\subset E\subset\mathcal{H}$ and $g:E\to\mathcal{H}$ is a 1-Lipschitz extension of f. This set is ordered in the natural fashion, and thus contains a maximal chain $\mathcal{C}$ by the Hausdorff maximality principle.

Define F to be the union of all E where $(E,g)\in\mathcal{C}$, and $h:F\to\mathcal{H}$ by $h(\xi)=g(\xi)$ if $\xi\in E$ and $(E,g)\in\mathcal{C}$. h is a well-defined 1-Lipschitz map because $\mathcal{C}$ is a chain. Clearly, h does not have a true extension to a 1-Lipschitz map. Hence, we conclude that $F=\mathcal{H}$ and h is the desired extension of f. □

Remark 3.1.4. Our presentation of the proof of the Kirszbraun-Valentine theorem follows [RS05] closely.

A minor but nonetheless useful Lipschitz extendibility result is as follows.

Proposition 3.1.5. Let (X,d) be a metric space, $\varnothing\neq D\subset X$, and $f:D\to\ell^\infty$ be Lipschitz. Then

$$g(x)_j=\inf_{y\in D}\big(f(y)_j+\mathrm{Lip}(f_j)\cdot d(x,y)\big)\quad\text{for all }\ x\in X\,,\ j\in\mathbb{N}$$

defines a Lipschitz extension $g:X\to\ell^\infty$ of f such that $\mathrm{Lip}(g)=\mathrm{Lip}(f)$.

Proof. Clearly, it suffices to argue for each component separately, so we reduce to the case that f takes values in $\mathbb{R}$. Let $x_1,x_2\in X$, and $\varepsilon>0$. There exists $y\in D$ such that $g(x_2)+\varepsilon\geqslant f(y)+\mathrm{Lip}(f)\cdot d(x_2,y)$. Thus

$$\begin{aligned}g(x_1)-g(x_2)&\leqslant\varepsilon+f(y)+\mathrm{Lip}(f)\cdot d(x_1,y)-\big(f(y)+\mathrm{Lip}(f)\cdot d(x_2,y)\big)\\&\leqslant\varepsilon+\mathrm{Lip}(f)\cdot d(x_1,x_2)\,.\end{aligned}$$

Since the right hand side is invariant under permutation of x_1 and x_2, it follows that g

is $\mathrm{Lip}(f)$-Lipschitz. Since g clearly extends f, we have $\mathrm{Lip}(g) = \mathrm{Lip}(f)$. $\square$

3.1.6. We point out that the previous result furnishes extensions (with possibly larger codomain) for any Lipschitz function with values in a separable metric space, since any such space can be embedded into ℓ^∞.

Indeed, let (X, d) be metric and (x_k) a dense sequence. We may assume d be bounded (otherwise consider, e.g., $\frac{d}{1+d}$). Define $\varphi : X \to \ell^\infty$ by $\varphi(x)_j = d(x, x_j) - d(x_0, x_j)$. Then

$$\left|d(x, x_j) - d(x_0, x_j) - \big(d(y, x_j) - d(x_0, x_j)\big)\right| = \left|d(x, x_j) - d(y, x_j)\right| \leqslant d(x, y) .$$

On the other hand, given $\varepsilon > 0$, there is $j \in \mathbb{N}$ such that $d(x, x_j) \leqslant \frac{\varepsilon}{2}$, so

$$\|\varphi(x) - \varphi(y)\|_\infty \geqslant \left|d(x, x_j) - d(y, x_j)\right| \geqslant d(y, x_j) - \frac{\varepsilon}{2} \geqslant d(x, y) - \varepsilon ,$$

and thus φ is indeed an isometry.

3.1.7. We have seen above that any separable metric space can be embedded in the unit ball of some dual Banach space. In this context we also mention that the converse can be easily established as follows. Let X be a separable Banach space. Choose a dense sequence (x_k) in $\mathbb{B}(X)$. Then

$$d_\sigma(\mu, \nu) = \sum_{k=0}^{\infty} \frac{1}{2^k} \cdot \left|\langle x_k : \mu - \nu\rangle\right| \quad \text{for all } \mu, \nu \in X^*$$

defines a metric d_σ on X which induces the $\sigma(X^*, X)$-topology on bounded subsets.

3.2 Differentiability of Functions of Bounded Variation

On our way to substantial multivariate differentiability results for Lipschitz functions, we need first to consider the one-variable case. The fundamental result in this domain is the following version of the Lebesgue differentiation theorem.

Theorem (Lebesgue) 3.2.1. Let $a, b \in \mathbb{R}$, $a < b$, and $f : [a, b] \to \mathbb{R}$ be increasing. Then f is $\mathcal{L}^1$ a.e. differentiable, the differential $f' : [a, b] \to \mathbb{R}$ is $\mathcal{L}^1$ summable, and

$$\int_a^b f' \, d\mathcal{L}^1 \leqslant f(b) - f(a) .$$

Proof. Define a finite measure μ on $\mathbb{R}$ by

$$\mu(A) = \inf\big\{f(d) - f(c) \;\big|\; A \cap [a, b] \subset [c, d] \subset [a, b]\big\} \quad \text{for all } A \subset \mathbb{R} .$$

It is obvious, by Caratheodory's criterion theorem 1.1.10, that μ is a Borel measure. By definition, for all $A \subset \mathbb{R}$, $\varepsilon > 0$, there exists an interval $A \subset [c, d] \subset [a, b]$ such that $\mu([c, d]) \leqslant \mu(A) + \varepsilon$. Hence, μ is Borel regular.

By proposition 1.9.3, $D_{\mathcal{L}^1}\mu(x)$ exists and is finite for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$. Hence, for $\mathcal{L}^1$ a.e. $x \in [a,b]$,

$$\frac{f(x+\varepsilon)-f(x)}{\varepsilon} + \frac{f(x)-f(x-\varepsilon)}{\varepsilon} = \frac{f(x+\varepsilon)-f(x-\varepsilon)}{\varepsilon} = 2 \cdot \frac{\mu\big(B(x,\varepsilon)\big)}{\mathcal{L}^1\big(B(x,\varepsilon)\big)}$$

converges to $D_{\mathcal{L}^1}\mu(x)$. Denote

$$D^+f(x) = \limsup_{\varepsilon\to 0+} \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \quad , \qquad D_+f(x) = \liminf_{\varepsilon\to 0+} \frac{f(x+\varepsilon)-f(x)}{\varepsilon} \, ,$$
$$D^-f(x) = \limsup_{\varepsilon\to 0+} \frac{f(x)-f(x-\varepsilon)}{\varepsilon} \quad , \qquad D_-f(x) = \liminf_{\varepsilon\to 0+} \frac{f(x)-f(x-\varepsilon)}{\varepsilon} \, .$$

If the left hand side of the above equation converges and one of its summands converges, the other also converges. This argument shows that

$$-\infty < D^+f(x) \leqslant D_{\mathcal{L}^1}\mu(x) \leqslant D_-f(x) < \infty \quad \text{for } \mathcal{L}^1 \text{ a.e. } x \in [a,b] \, .$$

The same reasoning, applied to $-f(a+b-\llcorner\!\lrcorner)$, gives $D^-f \leqslant D_+f$ $\mathcal{L}^1$ a.e. Hence,

$$-\infty < D^+f \leqslant D_{\mathcal{L}^1}\mu \leqslant D_-f \leqslant D^-f \leqslant D_+f \leqslant D^+f \quad \mathcal{L}^1 \text{ a.e.}$$

Therefore, f is $\mathcal{L}^1$ a.e. differentiable, and $f' = D_{\mathcal{L}^1}\mu$ $\mathcal{L}^1$ a.e., so theorem 1.9.5 gives

$$\int_a^b f' \, d\mathcal{L}^1 = \int_a^b D_{\mathcal{L}^1}\mu \, d\mathcal{L}^1 \leqslant \mu\big([a,b]\big) = f(b) - f(a) \, ,$$

Finally, f' is $\mathcal{L}^1$ measurable by proposition 1.9.3, and thus $\mathcal{L}^1$ summable. □

Definition 3.2.2. A function $f : [a,b] \to \mathbb{R}$ is said to have *bounded variation* if its total variation

$$\operatorname{Var}(f) = \sup_{a=x_0<\cdots<x_m=b} \sum_{j=0}^{m-1} \big|f(x_{j+1}) - f(x_j)\big|$$

is finite. Clearly, any Lipschitz function $f : [a,b] \to \mathbb{R}$ is of bounded variation with $\operatorname{Var}(f) \leqslant \operatorname{Lip}(f)$. (More generally, any absolutely continuous function is of bounded variation.) If f is defined on some (non-compact) interval, it is said to *locally have bounded variation* if $f \mid [a,b]$ is of bounded variation for any $[a,b] \subset I$.

The positive and negative variation are defined by

$$\operatorname{Var}^{\pm}(f) = \sup_{a=x_0<\cdots<x_m=b} \sum_{j=0}^{m-1} \big(f(x_{j+1}) - f(x_j)\big)^{\pm}$$

where we recall $x^+ = \max(x,0)$ and $x^- = -\min(x,0) = \max(-x,0)$ for all $x \in \mathbb{R}$. Observe that $x^{\pm} \geqslant 0$ and $x = x^+ - x^-$.

Proposition 3.2.3. Let $a \in I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ have locally bounded variation, and define, for all $x \in I$,

$$t^{\pm}(x) = \begin{cases} \mathrm{Var}^{\pm}(f \mid [a,x]) & a \leqslant x\,, \\ -\mathrm{Var}^{\pm}(f \mid [x,a]) & x < a\,. \end{cases}$$

Then $t^{\pm}$ are increasing, $f = f(a) + t^{+} - t^{-}$, and whenever $f = s^{+} - s^{-}$ where $s^{\pm}$ are increasing, then so are $s^{\pm} - t^{\pm}$.

Proof. Since f has bounded variation, $t^{\pm}$ are well-defined as real-valued functions. Let $a \leqslant x \leqslant y$. We have

$$\mathrm{Var}^{\pm}(f \mid [a,y]) - \mathrm{Var}^{\pm}(f \mid [a,x]) = \mathrm{Var}^{\pm}(f \mid [x,y]) \geqslant 0\,,$$

since we might as well take weakly increasing partitions in the definitions of the positive and negative variation. Hence, $t^{\pm}(y) - t^{\pm}(a) \geqslant 0$. By applying the permutation $(a,x,y) \mapsto (x,y,a)$, we find that this also true in case $x \leqslant y \leqslant a$. The case $x \leqslant a \leqslant y$ being trivial, $t^{\pm}$ are increasing.

Let $a = x_0 < \cdots < x_m = x$, and denote $\Sigma^{\pm} = \sum_{j=0}^{m-1}\bigl(f(x_{j+1}) - f(x_j)\bigr)^{\pm}$. Then

$$f(x) - f(a) = \sum_{j=0}^{m-1}\bigl(f(x_{j+1}) - f(x_j)\bigr) = \Sigma^{+} - \Sigma^{-} \in \bigl[\Sigma^{+} - t^{-}(x), t^{+}(x) - \Sigma^{-}\bigr]\,.$$

Noting $t^{+}(x) = \sup \Sigma^{+}$ and $-t^{-}(x) = \inf\bigl(-\Sigma^{-}\bigr)$, the supremum/infimum being taken over all partitions of $[a,x]$, we find $f(x) - f(a) = t^{+}(x) - t^{-}(x)$, as required. Exchanging x and a, we find that the equation is valid on all of I.

Finally, let $f = s^{+} - s^{-}$, the difference of increasing functions. Fix $x < y$ in I. Then

$$\begin{aligned} s^{+}(y) - t^{+}(y) &= f(y) - t^{+}(y) + s^{-}(y) = f(a) + t^{-}(y) + s^{-}(y) \\ &\geqslant f(a) + t^{-}(x) + s^{-}(x) = f(x) - t^{+}(x) + s^{-}(x) = s^{+}(x) - t^{+}(x)\,, \end{aligned}$$

so $s^{+} - t^{+}$ is increasing. Replacing f by $-f(a+b-\sqcup)$, we find that $s^{-} - t^{-}$ is increasing, too. $\square$

Remark 3.2.4. The above decomposition is often called the *minimal decomposition* of a function of locally bounded variation.

Corollary 3.2.5. Let $f : I \to \mathbb{R}$ be locally of bounded variation. Then f is $\mathcal{L}^1$ a.e. differentiable, and the derivative f' is locally $\mathcal{L}^1$ integrable. In particular, any locally Lipschitz function is $\mathcal{L}^1$ a.e. differentiable.

Proof. This is immediate from theorem 3.2.1, proposition 3.2.3, and the σ-compactness of $\mathbb{R}$. $\square$

Remark 3.2.6. Our presentation follows [Els99]. See, e.g., [Sim96] for an alternative approach.

3.3 Rademacher's Theorem, Weak and Metric Differentiability

The basic theorem on differentiability of is the following one, due to Rademacher. The Kirszbraun-Valentine extension theorem allows its application to any Lipschitz map defined on an arbitrary subset of $\mathbb{R}^m$.

Theorem (Rademacher) 3.3.1. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz, it is $\mathcal{L}^m$ a.e. differentiable.

Proof. Of course, if suffices to consider the case $n = 1$. Fix $e \in \mathbb{S}^{m-1}$. The set N_e of all $x \in \mathbb{R}^m$ where $t \mapsto f(x + t \cdot e)$ is not differentiable, is a Borel set. By corollary 3.2.5, $\mathcal{H}^1\big(N_e \cap (x + \mathbb{R}e)\big) = 0$ for all $x \in \mathbb{R}^m$. Thus, Fubini's theorem 1.6.2 allows us to conclude that $\mathcal{L}^m(N_e) = 0$.

Define the gradient $\nabla f(x) = \big(\partial_1 f(x), \ldots, \partial_m f(x)\big)$ whenever it exists. As we have seen, $\nabla f(x)$ exists for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$. Moreover, $\partial_e f(x)$ exists for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$. Let $\varphi \in \mathcal{C}_c^{(\infty)}(\mathbb{R}^m)$. For any $\varepsilon > 0$,

$$\int \frac{f(x + \varepsilon \cdot e) - f(x)}{\varepsilon} \cdot \varphi(x)\, d\mathcal{L}^m(x) = -\int f(x) \cdot \frac{\varphi(x) - \varphi(x - \varepsilon \cdot e)}{\varepsilon}\, d\mathcal{L}^m(x)\,.$$

Since $|\varphi| \leqslant \|\varphi\|_\infty \cdot 1_{\operatorname{supp}\varphi}$ and f is Lipschitz, we may apply Lebesgue's dominated convergence theorem 1.5.7 to the effect that

$$\begin{aligned}\int \partial_e f(x)\varphi(x)\, d\mathcal{L}^m(x) &= -\int f(x)\partial_e\varphi(x)\, d\mathcal{L}^m(x) = -\sum_{j=1}^m e_j \int f(x)\partial_j\varphi(x)\, d\mathcal{L}^m(x)\\ &= \sum_{j=1}^m e_j \int \partial_j f(x)\varphi(x)\, d\mathcal{L}^m(x) = \int (e : \nabla f(x))\varphi(x)\, dx\,.\end{aligned}$$

We conclude that $\partial_e f(x) = (e : \nabla f(x))$ for $\mathcal{L}^m$ a.e. x , for any $e \in \mathbb{S}^{m-1}$.

Fix a dense countable subset $E \subset \mathbb{S}^{m-1}$. Then $\mathcal{L}^m$ is concentrated on the intersection $A = \bigcap_{e \in E}\big\{\partial_e f = (e : \nabla f(x))\big\}$. We claim that f is differentiable at any $x \in A$. To that end, fix $x \in A$ and define

$$\Delta_\varepsilon(e) = \frac{f(x + \varepsilon \cdot e) - f(x)}{\varepsilon} - (e : \nabla f(x)) \quad \text{for all } \varepsilon > 0\,,\, e \in \mathbb{S}^{m-1}\,.$$

Suffices to prove $\lim_{\varepsilon \to 0+} \Delta_\varepsilon = 0$ uniformly. Set $M = 2\max\big(\operatorname{Lip}(f), \|\nabla f(x)\|, 1\big)$, and observe

$$\big|\Delta_\varepsilon(e) - \Delta_\varepsilon(e')\big| \leqslant \varepsilon^{-1} \cdot \big|f(x + \varepsilon e) - f(x + \varepsilon e')\big| + \|\nabla f(x)\| \cdot \|e - e'\| \leqslant M \cdot \|e - e'\|$$

for all $e, e' \in \mathbb{S}^{m-1}$, $\varepsilon > 0$.

Since $\mathbb{S}^{m-1}$ is compact, there exists a finite subset $E' \subset E$ such that $\operatorname{dist}(e, E') \leqslant \frac{\varepsilon}{2M}$ for all $e \in \mathbb{S}^{m-1}$. Moreover, $\lim_{\varepsilon \to 0+} \Delta_\varepsilon(e) = 0$ for all $e \in E'$. Hence, there is $\delta > 0$ such that $|\Delta_\varepsilon(e)| \leqslant \frac{\varepsilon}{2}$ for all $e \in E'$ and $0 < \varepsilon \leqslant \delta$. For $e \in \mathbb{S}^{m-1}$, $0 < \varepsilon \leqslant \delta$, we have

$$|\Delta_\varepsilon(e)| \leqslant \min_{e' \in E'} \big[|\Delta_\varepsilon(e) - \Delta_\varepsilon(e')| + |\Delta_\varepsilon(e')|\big] \leqslant \min_{e' \in E'} M \cdot \|e - e'\| + \frac{\varepsilon}{2} \leqslant \varepsilon\,,$$

so $\lim_{\varepsilon \to 0+} \Delta_\varepsilon = 0$ uniformly, and f is differentiable at x. □

Definition 3.3.2. Let $(X, \|\llcorner\!\lrcorner\|)$ be a normed space, (Y, d) a metric space, $U \subset X$ an open subset, and $f : U \to Y$. If $x \in U$ and there exists a continuous seminorm $p : X \to \mathbb{R}$ such that

$$\lim_{y \to x} \frac{1}{\|y - x\|} \cdot \big[d(f(y), f(x)) - p(y - x)\big] = 0\,,$$

then f said to be *metrically differentiable* at x. We point out that the continuity of p is automatic if X is finite-dimensional.

Lemma 3.3.3. If $f : U \to Y$ and $x \in U$, then there is at most one seminorm p as in definition 3.3.2. If it exists, we denote it by $|f'|(x)$. If f is metrically differentiable at x, then it is continuous at x.

Proof. Let $p : X \to \mathbb{R}$ fulfill the condition. Then fix $y \in X$. For all $\varepsilon > 0$,

$$p(y) = \|y\| \cdot \frac{p(x + \varepsilon y - x)}{\|x + \varepsilon y - x\|}\,,$$

and therefore

$$p(y) = \lim_{\varepsilon \to 0+} \frac{d\big(f(x + \varepsilon y), f(x)\big)}{\varepsilon}\,.$$

Thus follows the uniqueness. The statement about continuity is trivial. □

Metric differentiability is closely related to the notion of weak differentiability.

Definition 3.3.4. Let $(X, \|\llcorner\!\lrcorner\|)$ be a normed space, $Y = E^*$ a dual Banach space, $U \subset X$ an open subset, and $f : U \to Y$. If $x \in U$ and there exists a continuous linear map $L : X \to Y_\sigma$ such that

$$\lim_{y \to x} \frac{1}{\|y - x\|} \cdot \big[f(y) - f(x) - L(y - x)\big] = 0 \quad \text{in} \quad \sigma(Y, E)\,,$$

then f is said to be *weak* differentiable* at x. The continuity of L is automatic if X is finite-dimensional. Moreover, as above, L is unique if it exists, in which case we denote it by $f'_\sigma(x)$. If f is weakly differentiable at x, then $f : U \to Y_\sigma$ is continuous at x.

3.3.5. Let $f : U \to Y = E^*$ be weakly and metrically differentiable at $x \in U$, where we the metric d induced by the dual norm. We claim that

$$\|f'_\sigma(x)v\| \leqslant |f'|(x)v \quad \text{for all} \ v \in X\,.$$

Indeed, since $\|y\| = \sup_{\|e\|\leqslant 1}|\langle e : y\rangle|$ for all $y \in Y$, $\|\sqcup\|$ is $\sigma(Y,E)$-l.s.c. Thus

$$\|f'_\sigma(x)v\| = \left\|\lim_{\varepsilon\to 0+} \frac{f(x+\varepsilon v)-f(x)}{\|x+\varepsilon v - x\|}\right\| \leqslant \liminf_{\varepsilon\to 0+} \frac{\|f(x+\varepsilon v)-f(x)\|}{\varepsilon} = |f'|(x)v$$

for all $v \in X$, $\|v\| = 1$. The positive homogeneity of both sides of the inequality ensues our statement.

The following theorem shows that this inequality is a.e. an equality.

Theorem (Ambrosio-Kirchheim) 3.3.6. Let E be a separable Banach space, $Y = E^*$, and $f : \mathbb{R}^m \to Y$ be Lipschitz. Then for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$, f is weakly and metrically differentiable (w.r.t. the dual norm), and $\|f'_\sigma(x)\| = |f'|(x)$.

Proof. Let $D \subset E$ be a dense countable-dimensional $\mathbb{Q}$-subspace. By theorem 3.3.1, the set $N \subset \mathbb{R}^m$ of all $x \in \mathbb{R}$ such that $\langle \xi : f\rangle$ is not differentiable at x for some $\xi \in D$ is $\mathcal{L}^m$ negligible. Let $A = \mathbb{R}^m \setminus N$ and note

$$\big|\langle \xi : f\rangle'(x)v\big| = \lim_{\varepsilon\to 0+} \frac{\big|\langle \xi : f(x+\varepsilon v)-f(x)\rangle\big|}{\varepsilon} \leqslant \operatorname{Lip}(f)\cdot\|\xi\|\cdot\|v\|,$$

so $\mathbb{R}^m \times D \to \mathbb{R} : (v,\xi) \mapsto \langle \xi : f\rangle'(x)v$ is uniformly continuous for all $x \in A$. Thus, there exists a continuous linear map $\nabla f(x) : \mathbb{R}^m \to Y$ so that

$$\langle \xi : \nabla f(x)v\rangle = \langle \xi : f\rangle'(x)v \quad \text{for all } v \in \mathbb{R}^m,\ \xi \in D.$$

By exactly the same argument as in the proof of Rademacher's theorem, f is weakly differentiable at every $x \in A$, and $\nabla f(x) = f'_\sigma(x)$. Moreover, as above, we have

$$\|f'_\sigma(x)v\| \leqslant \liminf_{\varepsilon\to 0+} \frac{\|f(x+\varepsilon v)-f(x)\|}{\varepsilon} \quad \text{for all } v \in \mathbb{R}^m,\ x \in A.$$

It is clear that

$$\langle \xi : \nabla f(x)v\rangle = \lim_{\varepsilon\to 0+} \frac{\langle \xi : f(x+\varepsilon v)-f(x)\rangle}{\varepsilon} \quad \text{for all } x \in A,\ v \in \mathbb{R}^m,\ \xi \in D.$$

Define $\nabla f = 0$ on N. Let $x \in A$, $v \in \mathbb{R}^m$, $e \in E$ and $\xi \in D$. Then

$$\begin{aligned}&\int_0^t \frac{\langle \xi : f(x+(\varepsilon+\tau)v)-f(x+\tau v)\rangle}{\varepsilon}\,d\tau \\ &\quad = \frac{1}{\varepsilon}\cdot\Big[\int_t^{t+\varepsilon}\langle \xi : f(x+\tau v)\rangle\,d\tau - \int_0^\varepsilon \langle \xi : f\rangle(x+\tau v)\,d\tau\Big] \to \langle \xi : f(x+tv)-f(x)\rangle,\end{aligned}$$

as follows immediately from the continuity of f. Moreover, the integrand on the left hand side has an integrable bound (f is Lipschitz), so we may apply Lebesgue's dominated convergence theorem 1.5.7 to achieve

$$\langle \xi : f(x+tv) - f(x) \rangle = \int_0^t \langle \xi : \nabla f(x+\tau v) v \rangle \, d\tau \,,$$

since $x + \tau v \in A$ for a.e. $\tau \in [0,t]$. In particular,

$$\limsup_{t \to 0+} \frac{\| f(x+tv) - f(x) \|}{t} \leqslant \lim_{t \to 0+} \int_0^t \| \nabla f(x+\tau v) v \| \, d\tau$$

By the Lebesgue-Besicovich differentiation theorem 1.9.9, shrinking A , we may assume the right hand side is $\|\nabla f(x) v\| = \|f'_\sigma(x) v\|$, whence our claim. □

This result furnishes us with the following striking extension of Rademacher's theorem.

Theorem (Rademacher-Ambrosio-Kirchheim) 3.3.7. Let $f : \mathbb{R}^n \to X$ be Lipschitz where (X,d) is metric. Then f is $\mathcal{L}^n$ a.e. metrically differentiable.

Proof. Since $f(\mathbb{R}^n)$ is separable, we may assume that this is the case with X , too. Thus X may be considered as a subspace of $\ell^\infty = (\ell^1)^*$. Now, theorem 3.3.6 applies. □

In fact, we can give a version of the mean value theorem. To that end, we give the following definition.

Definition 3.3.8. Let X , Y be metric spaces and $f : X \to Y$. An continuous increasing function $\omega : [0,\infty[\to [0,\infty[$ such that $\omega(0) = 0$, $\omega(x+y) \leqslant \omega(x) + \omega(y)$ for all $x, y \geqslant 0$, and

$$d\big(f(x), f(y)\big) \leqslant \omega\big(d(x,y)\big) \quad \text{for all } \; x, y \in X$$

is called a (global) *modulus of continuity* for f . Then f has a modulus of continuity if and only if it is continuous.

Moreover, if E is normed, then on the set of seminorms on E , we introduce the metric

$$\delta(p,q) = \sup_{\|x\| \leqslant 1} |p(x) - q(x)| \quad \text{for all seminorms } p, q \text{ on } E \,.$$

We note that $\delta(p,q) \leqslant \varepsilon$ is equivalent to $|p(x) - q(x)| \leqslant \varepsilon \|x\|$ for all $x \in E$.

Metric Mean Value Inequality (Kirchheim, Ambrosio-Kirchheim) 3.3.9. For any Lipschitz $f : \mathbb{R}^n \to X$, (X,d) metric,

$$\lim_{x \neq y, z \to 0} \frac{d\big(f(y), f(z)\big) - |f'|(x)(y-z)}{\|y - x\| + \|x - z\|} = 0 \quad \text{for } \mathcal{L}^n \text{ a.e. } x \in \mathbb{R}^n \,.$$

Moreover, there exists a countable compact family $\mathcal{K}$ such that $|f'| \,\big|\, K$ is continuous for each $K \in \mathcal{K}$, and moduli of continuity for $|f'| \,\big|\, K$, ω_K , such that

$$\big| d\big(f(y), f(z)\big) - |f'|(z)(y-z) \big| \leqslant \omega_K\big(\|y-z\|\big) \cdot \|y-z\| \quad \text{for all } \; y \in \mathbb{R}^n \,,\, z \in K \in \mathcal{K} \,.$$

Proof. By theorem 3.3.7, for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$, $|f'|(x)$ exists. For each $y \in \mathbb{R}^n$, $|f'|(\sqcup)(y)$ is $\mathcal{L}^n$ measurable. Since the unit ball of $\mathbb{R}^n$ is separable, it follows easily that $|f'|$ is $\mathcal{L}^n$ measurable. By Lusin's theorem 1.4.1, there exist a countable compact $\mathcal{L}^n$ almost cover $\mathcal{K}$ of $\mathbb{R}^n$, and such that $|f'| \mid K$ is continuous for all $K \in \mathcal{K}$. Moreover, by Egorov's theorem 1.4.4, we may assume $\lim_{r \to 0+} \frac{1}{r} \cdot d(f(x), f(x+ry)) = |f'|(x)y$ uniformly in $x \in K$ and $y \in B(0,1)$.

By corollary 1.9.10, we need only prove the assertion at points x of $\mathcal{L}^n$ density 1 for some $K \in \mathcal{K}$. Fix a density point $x \in K$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$B(x+rv, r\varepsilon) \cap K \neq \varnothing \quad \text{for all } \|v\| \leqslant \varepsilon^{-1}, 0 < r \leqslant \delta. \tag{$*$}$$

By the uniformity of the above convergence, we may assume

$$\left| \frac{d\big(f(y+rv), f(y)\big)}{r} - |f'|(y)v \right| \leqslant \varepsilon^2 \quad \text{for all } y \in K, v \in B(0,1), 0 < r \leqslant \frac{2\delta}{\varepsilon}. \tag{$**$}$$

Since $|f'| \mid K$ is continuous, we may also assume

$$\delta\big(|f'|(x), |f'|(y)\big) \leqslant \varepsilon^2 \quad \text{for all } y \in K, \|x-y\| \leqslant \delta\big(\varepsilon + \tfrac{1}{\varepsilon}\big). \tag{$***$}$$

For $u, v \in B\big(0, \frac{1}{\varepsilon}\big)$, $u \neq v$, there exists $w \in K$, $\|w - (x+rv)\| \leqslant \varepsilon r \leqslant \delta\big(\varepsilon + \frac{1}{\varepsilon}\big)$. Thus,

$$\begin{aligned}
&\left| \frac{1}{r} \cdot d\big(f(x+ru), f(x+rv)\big) - |f'|(x)(u-v) \right| \\
&\quad \leqslant \left| \frac{d\big(f(w+r(u-v)), f(w)\big)}{r} - |f'|(x)(u-v) \right| \\
&\qquad + \frac{1}{r} \cdot \big[d\big(f(x+ru), f(w+r(u-v))\big) + d\big(f(w), f(x+\varepsilon v)\big) \big];
\end{aligned}$$

now, we set $z = \frac{u-v}{\|u-v\|}$, and hence,

$$\begin{aligned}
&\leqslant \|u-v\| \cdot \left[\left| \frac{d\big(f(w+r\|u-v\|z), f(w)\big)}{r\|u-v\|} - |f'|(w)z \right| + \big| |f'|(x)z - |f'|(w)z \big| \right] \\
&\quad + \frac{2 \cdot \mathrm{Lip}(f)}{r} \cdot \|w - (x+rv)\| \\
&\leqslant 2\varepsilon^2 \cdot \|u-v\| + 2\varepsilon \cdot \mathrm{Lip}(f) \leqslant 2\big(2 + \mathrm{Lip}(f)\big)\varepsilon.
\end{aligned}$$

This gives the first assertion, setting $y = x + ru$, $z = y + rv$, since then $r = \|x-y\| = \|x-z\|$. The condition $(**)$ gives

$$\big| d\big(f(y+\delta v), f(y)\big) - |f'|(y)(y+\delta v - y) \big| \leqslant \varepsilon^2 \cdot \delta.$$

By $(***)$, $\omega_K(\delta) = \varepsilon^2$ is modulus of continuity for $|f'||K$, hence the assertion. — □

3.3.10. The set of norms on $\mathbb{R}^n$, endowed with the metric δ, is a separable metric space. In fact, the set of all finite subsets of $\mathbb{Q}^n$ is countable. Let p be a norm on $\mathbb{R}^n$, and $\varepsilon > 0$. We assume, as we may, that $r \cdot \|\llcorner\!\lrcorner\|_2 \leqslant p \leqslant \|\llcorner\!\lrcorner\|_2$. Choose $x_0, \dots, x_N \in \mathbb{Q}^n$, such that $\{p = 1\} \subset \bigcup_{j=0}^N B(x_j, \varepsilon)$. Let $B = \mathrm{co}(\pm x_0, \dots, \pm x_N)$. Then B is convex and symmetric, so the Minkowski gauge

$$q(x) = \inf\bigl\{t > 0 \bigm| t^{-1}x \in B\bigr\} \quad \text{for all } \ x \in \mathbb{R}^n$$

is a norm on $\mathbb{R}^n$. By supplementing $\pm x_j$ by rational multiples of standard basis elements e_j, we may assume $co(\pm e_1, \dots, \pm e_n) \subset B$, so $q \leqslant \sqrt{n}\|\llcorner\!\lrcorner\|_2$. (Note $p(\pm e_j) \leqslant 1$.) Let $0 < \|x\| \leqslant 1$. Then for all $j = 0, \dots, N$,

$$|p(x) - q(x)| = p(x) \cdot \left|1 - q\left(\frac{x}{p(x)}\right)\right| = p(x) \cdot \left|q(x_j) - q\left(\frac{x}{p(x)}\right)\right| \leqslant \sqrt{n} \cdot \left\|x_j - \frac{x}{p(x)}\right\| .$$

Since $\mathrm{dist}(\{x_0, \dots, x_N\}, \{p = 1\}) \leqslant \frac{\varepsilon}{r}$, we find $\delta(p, q) \leqslant \frac{\sqrt{n}\varepsilon}{r}$.

Proposition 3.3.11. Let $f : \mathbb{R}^n \to X$ be Lipschitz, $\lambda > 1$. There exist disjoint Borel sets $B_j \subset \mathbb{R}^n$, such that $\bigcup_{j=0}^\infty B_j$ is the set of points at which f is metrically differentiable and $|f'|$ is a norm, and norms p_j on $\mathbb{R}^n$, such that

$$\frac{1}{\lambda} p_j(x - y) \leqslant d\bigl(f(x), f(y)\bigr) \leqslant \lambda \cdot p_j(x - y) \quad \text{for all } \ x, y \in B_j .$$

Proof. Let P be a dense countable subset of the set of all norms on $\mathbb{R}^n$, and fix $\varepsilon > 0$, such that $\frac{1}{\lambda} + \varepsilon < 1 < \lambda - \varepsilon$. Let B be the Borel set of points at which f is metrically differentiable. Then for $x \in B$, $|f'|(x)$ is a norm if and only if $x \in \bigcup_{p \in P} B_p$ where

$$B_p = \bigl\{y \in B \bigm| \forall\, v \in \mathbb{R}^n \ : \ \bigl(\tfrac{1}{\lambda} + \varepsilon\bigr)p(v) \leqslant |f'|(y)v \leqslant (\lambda + \varepsilon)p(v)\bigr\} .$$

Moreover, by the definition of metric differentiability, the Borel sets

$$B_{p,k} = \bigl\{x \in B_p \bigm| \forall\, y \in B\bigl(x, \tfrac{1}{k}\bigr) \ : \ \bigl|d(f(x), f(y)) - |f'|(x)(x - y)\bigr| \leqslant \varepsilon \cdot p(x - y)\bigr\}$$

for all $p \in P$, $k \in \mathbb{N} \setminus 0$, form a cover of B. For each $(p, k) \in P \times (\mathbb{N} \setminus 0)$, form a countable disjoint Borel partition by sets $B_{pk\ell}$ of diameter $\leqslant \frac{1}{k}$. Then, for $x, y \in B_{pk\ell}$,

$$\tfrac{1}{\lambda} p(x - y) \leqslant |f'|(x)(x - y) - \varepsilon p(x - y) \leqslant d\bigl(f(x), f(y)\bigr) \leqslant \lambda p(x - y) ,$$

proving the assertion. □

Remark 3.3.12. Our presentation of Rademacher's theorem 3.3.1 follows [Mat95], the remainder of the subsection follows [AK00b] and [Kir94]

4 Rectifiability

4.1 Area Formula

Definition 4.1.1. Let $0 \leqslant s < \infty$, V, W be a normed vector spaces and $p : V \to [0,\infty]$ a sublinear functional. Define the *s-Jacobian of p* by

$$\mathrm{J}_s(p) = \frac{\omega_s}{\mathcal{H}^s\{p \leqslant 1\}} \in [0,\infty] ,$$

Here, $\frac{c}{0} = \infty$ and $\frac{c}{\infty} = 0$ for $c > 0$, and the Hausdorff measure is defined w.r.t. the norm on V. If $L : V \to W$ is linear, define the *s-Jacobian of L* by

$$\mathrm{J}_s(L) = \mathrm{J}_s(\|L\|) .$$

Proposition 4.1.2. Let $k = \dim U = \dim V \leqslant \dim W$ where U, V and W are normed. Whenever $S : U \to V$ and $T : V \to W$ are linear,

$$\mathrm{J}_k(T \circ S) = \mathrm{J}_k(T) \cdot \mathrm{J}_k(S) .$$

Proof. Since $T(V)$ is contained in a k-dimensional subspace of W, we may assume $\dim W = k$. Moreover, we may assume $U = V = W = \mathbb{R}^k$ with possibly distinct norms. Then for all $x \in \mathbb{R}^k$, and $r > 0$

$$\mathrm{J}_k(T) = \frac{\mathcal{H}^k\big(B_W(0,1)\big)}{\mathcal{H}^k\big(T^{-1}B_W(0,1)\big)} = \frac{\mathcal{H}^k\big(B_W(x,r)\big)}{\mathcal{H}^k\big(T^{-1}B_W(x,r)\big)} ,$$

by theorem 2.4.3. If $\mathcal{H}^k\big(T^{-1}(B_W(0,1))\big) = \infty$, then since $\mathcal{H}^k(A(\mathbb{R}^m)) = 0$ for any $m < k$ and any linear $A : \mathbb{R}^m \to \mathbb{R}^k$, we find $T^{-1}(B(0,1)) = \mathbb{R}^k$; in other words, $T = 0$. Otherwise, corollary 1.9.6 shows that $\mathrm{J}_k(T) \cdot T(\mathcal{H}^k) = \mathcal{H}^k$. Hence

$$\begin{aligned}\mathrm{J}_k(S \circ T) \cdot (S \circ T)(\mathcal{H}^k) = \mathcal{H}^k = \mathrm{J}_k(S) \cdot S(\mathcal{H}^k) = \mathrm{J}_k(S) \cdot S\big(\mathrm{J}_k(T) \cdot T(\mathcal{H}^k)\big) \\ = \mathrm{J}_k(S) \cdot \mathrm{J}_k(T) \cdot S(T(\mathcal{H}^k)) = \mathrm{J}_k(S) \cdot \mathrm{J}_k(T) \cdot (S \circ T)(\mathcal{H}^k) .\end{aligned}$$

This proves the proposition. □

Area Formula 4.1.3. Let (X,d) be metric, $f : \mathbb{R}^n \to X$ be Lipschitz, and $A \subset \mathbb{R}^n$ be Borel. Then

$$\int_A \mathrm{J}_n(|f'|(x))\, d\mathcal{L}^n(x) = \frac{\alpha_n}{\omega_n} \cdot \int_X N(f \mid A, y)\, d\mathcal{H}^n_d(y)$$

where $\alpha_n = \mathcal{L}^n\big(B(0,1)\big)$. Moreover, $\mathrm{J}_n(|f'|(x)) = 0$ for $\mathcal{L}^n$ a.e. point x at which $|f'|(x)$ is not a norm.

Remark 4.1.4. As a corollary, we shall prove $\alpha_n = \omega_n$.

We first note the following simple lemma.

Lemma 4.1.5. Let E be normed, $0 \leqslant s < t < \infty$ and $f : E \to]0,\infty[$ be positively homogeneous. Then

$$\lim_{\varepsilon\to 0+} \int_{\{\varepsilon \leqslant f < 1\}} f^{-t}\, d\mathcal{H}^s = \infty\,.$$

Proof. If $g \geqslant 0$ is any integrable simple function, then for all $\lambda > 0$,

$$\int g(\lambda x)\, d\mathcal{H}^s(x) = \sum_{0\leqslant y\leqslant\infty} y\mathcal{H}^s\big(\lambda^{-1}g^{-1}(y)\big) = \lambda^{-s}\cdot\int g(x)\, d\mathcal{H}^s(x)\,.$$

Hence, the corresponding formula is true for any positive function in place of g. Define sets $A_j = \{2^{-(j+1)} \leqslant f < 2^{-j}\}$. Then $A_j = 2A_{j+1}$ since f is positively homogeneous, and we deduce

$$\int_{A_j} f^{-t}\,\mathcal{H}^s = 2^{-t}\int 1_{A_{j+1}}\big(\tfrac{x}{2}\big) f\big(\tfrac{x}{2}\big)^{-t}\,\mathcal{H}^s(x) = 2^{s-t}\cdot\int_{A_{j+1}} f^{-t}\,\mathcal{H}^s\,.$$

Therefore,

$$\begin{aligned}\int_{\{2^{-j}\leqslant f<1\}} f^{-t}\, d\mathcal{H}^s &= \sum_{k=0}^{j-1}\int_{A_j} f^{-t}\, d\mathcal{H}^s\\ &= \int_{\{1/2\leqslant f<1\}} f^{-t}\, d\mathcal{H}^s\cdot\sum_{k=0}^{j-1}2^{k(t-s)} = \frac{2^{j(t-s)}-1}{2^{t-s}-1}\cdot\int_{\{1/2\leqslant f<1\}} f^{-t}\, d\mathcal{H}^s\,,\end{aligned}$$

which tends to infinity for $j \to \infty$. □

Proof of theorem 4.1.3. We first assume that for any $x \in A$, $|f'|(x)$ exists and $\mathrm{J}_n\big(|f'|(x)\big)$ is a norm. Let $\tau > 1$. By proposition 3.3.11, there exist countable Borel partitions P_j of A, $\lim_j \sup\{\operatorname{diam} B | B \in P_j\} = 0$, and norms $\|\sqcup\|_B$, $B \in P_j$, on $\mathbb{R}^n$ such that

$$\tau^{-1}\cdot\|x-y\|_B \leqslant d\big(f(x),f(y)\big) \leqslant \tau\cdot\|x-y\|_B \quad \text{for all } x,y\in B\in P_j\,.$$

This implies, on the one hand,

$$\frac{1}{\tau}\cdot\|y\|_B = \frac{1}{\tau r}\cdot\|x-(x+ry)\|_B \leqslant |f'|(x)y = \lim_{r\to 0+}\frac{d\big(f(x),f(x+ry)\big)}{r} \leqslant \tau\cdot\|y\|_B$$

whenever $x \in B$ is an $\mathcal{L}^n$ density point, and on the other hand,

$$\tau^{-n}\cdot\mathcal{H}^n_{\|\sqcup\|_B}(C\cap B) \leqslant \mathcal{H}^n_d\big(f(C\cap B)\big) \leqslant \tau^n\cdot\mathcal{H}^n_{\|\sqcup\|_B}(C\cap B) \quad \text{for all } C\subset A\,.$$

The first estimate gives

$$\mathrm{J}_n(\|\sqcup\|_B) = \frac{\omega_n}{\mathcal{H}^n\{\|\sqcup\|_B\leqslant 1\}} \geqslant \frac{\omega_n}{\mathcal{H}^n\{|f'|(x)\leqslant\tau\}} = \tau^{-n}\cdot\mathrm{J}_n\big(|f'|(x)\big) \quad \text{for } \mathcal{L}^n \text{ a.e. } x\in B$$

and similarly $\mathrm{J}_n(|f'|(x)) \geqslant \tau^{-n} \cdot \mathrm{J}_n(\|\llcorner\!\lrcorner\|_B)$ for $\mathcal{L}^n$ a.e. $x \in B$. Set $\alpha_n = \mathcal{L}^n(B(0,1))$; then

$$\begin{aligned}\tau^{-2n} \int_B \mathrm{J}_n(|f'|)\, d\mathcal{L}^n \leqslant \tau^{-n} \int_B \mathrm{J}_n(\|\llcorner\!\lrcorner\|_B)\, d\mathcal{L}^n &= \frac{\tau^{-n} \cdot \omega_n}{\mathcal{H}^n\{\|\llcorner\!\lrcorner\|_B \leqslant 1\}} \cdot \mathcal{L}^n(B) \\ &= \frac{\tau^{-n} \cdot \omega_n \cdot \mathcal{L}^n\{\|\llcorner\!\lrcorner\|_B \leqslant 1\}}{\mathcal{H}^n\{\|\llcorner\!\lrcorner\|_B \leqslant 1\} \cdot \omega_n} \cdot \mathcal{H}^n_{\|\llcorner\!\lrcorner\|_B}(B) = \frac{\tau^{-n} \cdot \alpha_n}{\omega_n} \cdot \mathcal{H}^n_{\|\llcorner\!\lrcorner\|_B}(B) \\ &\leqslant \frac{\alpha_n}{\omega_n} \cdot \mathcal{H}^n_d(f(B)) = \frac{\alpha_n}{\omega_n} \cdot \int 1_{f(B)}\, d\mathcal{H}^n_d \leqslant \frac{\tau^{2n} \cdot \alpha_n}{\omega_n} \cdot \int_B \mathrm{J}_n(|f'|)\, d\mathcal{L}^n ,\end{aligned}$$

where theorem 2.4.3 was employed. As in the proof of theorem 2.2.2, we find

$$g_j = \textstyle\sum_{B \in P_j} 1_{f(B)} \leqslant g_{j+1} \to N(f|A, \llcorner\!\lrcorner) ,$$

so corollary 1.5.6 gives the equation.

Since $|f'|(x)$ exists for a.e. $x \in \mathbb{R}^n$, and both sides of the equation equal zero for $\mathcal{L}^n$ negligible A , it remains to prove that both sides vanish for any A such that for each $x \in A$, $|f'|(x)$ exists, but is not a norm. To that end, factor $f = \pi \circ f_\varepsilon$ where $\varepsilon > 0$,

$$f_\varepsilon : \mathbb{R}^n \to X \times \mathbb{R}^n : x \mapsto (f(x), \varepsilon x) \quad \text{and} \quad \pi = \mathrm{pr}_1 : X \times \mathbb{R}^n \to X .$$

If we consider the box metric on $X \times \mathbb{R}^n$, then f_ε is $L = \mathrm{Lip}(f)$ Lipschitz for $\varepsilon \leqslant L$ and π is 1 Lipschitz. Hence,

$$\int N(f \mid B)\, d\mathcal{H}^n \leqslant \mathcal{H}^n(f_\varepsilon(B)) \quad \text{for all } B \in \mathcal{B}(X) ,\, B \subset A ,$$

by theorem 2.2.2. (Since $f(\mathbb{R}^n)$ is separable, there is no loss in generality to assume X separable.) Choosing Borel partitions as above, we find

$$\int N(f \mid A)\, d\mathcal{H}^n \leqslant \int N(f_\varepsilon \mid A)\, d\mathcal{H}^n = \frac{\omega_n}{\alpha_n} \cdot \int_A \mathrm{J}_n(|f'_\varepsilon|)\, d\mathcal{L}^n ,$$

where the first part is applicable since $|f'_\varepsilon|(x) = \max(|f'|(x), \varepsilon \cdot \|\llcorner\!\lrcorner\|_2)$ is a norm whenever it exists. Thus, $|f'| \leqslant |f'_\varepsilon|$. Hence, both sides of the equation vanish as soon as

$$\lim_{\varepsilon \to 0+} \mathrm{J}_n(|f'_\varepsilon|(x)) = 0 \quad \text{for all } x \in A .$$

To prove this statement, let $u \in \mathbb{S}^{n-1}$ be such that $|f'|(x)u = 0$. Then define

$$p_\pm : C = B(0,1) \cap u^\perp \to \mathbb{S}^{n-1} : y \mapsto y \pm \sqrt{1 - \|y\|^2} \cdot u \text{ so that } \mathbb{S}^{n-1} = p_+(C) \cup p_-(C)$$

with $\mathcal{H}^{n-1}(p_+(C) \cap p_-(C)) = 0$. Then

$$\|p_\pm(x) - p_\pm(y)\|^2 = \|x - y\|^2 + \left(\sqrt{1 - \|x\|^2} - \sqrt{1 - \|y\|^2}\right)^2 \geqslant \|x - y\|^2 ,$$

which implies $\mathcal{H}^{n-1}(p_\pm(B)) \geqslant \mathcal{H}^{n-1}(B)$ for all Borel sets $B \subset C$. Furthermore,

$$\begin{aligned} |f'|(x)\big(p_\pm(y)\big) &\leqslant |f'|(x)y + \sqrt{1-\|y\|^2} \cdot |f'|(x)u = |f'|(x)y \\ &\leqslant |f'|(x)\big(p_\pm(y)\big) + \sqrt{1-\|y\|^2} \cdot |f'|(x)u = |f'|(x)\big(p_\pm(y)\big) \,. \end{aligned}$$

Thus, $|f'|(x)\big(p_\pm(y)\big) = |f'|(x)y$. Because $\operatorname{Lip}(|f'_\varepsilon|) \leqslant \max(L, \varepsilon) = L$, we may estimate

$$\begin{aligned} \frac{2nL\omega_{n-1}}{\omega_n} \cdot \mathcal{H}^n\{|f'_\varepsilon|(x) \leqslant 1\} &\geqslant \int_{\mathbb{S}^{n-1}} [|f'|(x)y]^{-n}\, \mathcal{H}^{n-1}(y) \\ &= \sum_{\varepsilon^2=1} \int_{p_\varepsilon(C)} \min\big((|f'|(x)p_\varepsilon^{-1}(y))^{-n}, \varepsilon^{-n}\big)\, d\mathcal{H}^{n-1}(y) \\ &\geqslant 2\int_C \min\big((|f'|(x)y)^{-n}, \varepsilon^{-n}\big)\, d\mathcal{H}^{n-1}(y) \\ &\geqslant 2\int_C \min\big((L\|y\|)^{-n}, \varepsilon^{-n}\big)\, d\mathcal{H}^{n-1}(y) \geqslant \frac{2}{L^n} \int_{\{\frac{\varepsilon}{L} \leqslant \|\sqcup\| \leqslant 1\}} \|\sqcup\|^{-n}\, d\mathcal{H}^{n-1} \,, \end{aligned}$$

by corollary 2.4.6 and theorem 2.2.2. Hence, for some $R > 0$,

$$\begin{aligned} \lim_{\varepsilon\to 0+} \mathrm{J}_n\big(|f'_\varepsilon|(x)\big) &= \lim_{\varepsilon\to 0+} \frac{\omega_n}{\mathcal{H}^n\{|f'_\varepsilon|(x) \leqslant 1\}} \\ &\leqslant R \cdot \lim_{\varepsilon\to 0+} \Big(\int_{\{\varepsilon/L \leqslant \|\sqcup\| \leqslant 1\}} \|\sqcup\|^{-n}\, d\mathcal{H}^{n-1}\Big)^{-1} = 0 \,, \end{aligned}$$

by lemma 4.1.5, proving the theorem. □

We obtain the following rather general change of variables formula.

Corollary 4.1.6. Let (X, d) be metric, $f : \mathbb{R}^n \to X$ be Lipschitz, and consider $\mathcal{H}^n = \mathcal{H}^n_d$.

(i). If $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ is Borel, then

$$\int_{\mathbb{R}^n} g(x)\mathrm{J}_n\big(|f'|(x)\big)\, d\mathcal{L}^n(x) = \frac{\alpha_n}{\omega_n} \cdot \int_X \textstyle\sum_{f(x)=y} g(x)\, d\mathcal{H}^n(y) \,,$$

provided one the integrals exists.

(ii). If $h : X \to \bar{\mathbb{R}}$ and $A \subset \mathbb{R}$ are Borel, then

$$\int_A h(f(x))\mathrm{J}_n\big(|f'|(x)\big)\, d\mathcal{L}^n(x) = \frac{\alpha_n}{\omega_n} \cdot \int_X h(y)N(f \mid A, y)\, d\mathcal{H}^n(y) \,,$$

provided one of the integrals exists.

Proof of (i). First, let $g \geqslant 0$. By lemma 1.3.3, we obtain by pyramidal approximation a sequence (A_k) of Borel sets, and constants $a_k > 0$ such that $g = \sum_{k=0}^\infty a_k 1_{A_k}$. By corollary 1.5.6 and theorem 4.1.3,

$$\int_{\mathbb{R}^n} g \cdot \mathrm{J}_n\big(|f'|\big)\, d\mathcal{L}^n = \sum_{k=0}^\infty a_k \int_{A_k} \mathrm{J}_n\big(|f'|\big)\, d\mathcal{L}^n$$

$$= \frac{\alpha_n}{\omega_n} \cdot \sum_{k=0}^{\infty} a_k \int_X N(f \mid A_k, \llcorner)\, d\mathcal{H}^n = \frac{\alpha_n}{\omega_n} \cdot \int_X \sum\nolimits_{f(y)=x} g(x)\, d\mathcal{H}^n(y)\,,$$

since $\sum g\big(f^{-1}(y)\big) = \sum_{k=0}^{\infty} a_k \cdot \#\big(A_k \cap f^{-1}(y)\big)$. The general case now follows by definition of $\int$, writing $g = g^+ - g^-$.

Proof of (ii). This follows form (i), applied to the function $g = 1_A \cdot h \circ f$. ∎

In order to give some applications of these formulae, we need to find computable expressions for the Jacobian. We give some expressions in the Euclidean setup, which also link the above results to the more familiar Euclidean change of variables formula.

Proposition 4.1.7. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then $\mathrm{J}_n(L) = 0$ for $n > m$, and for $n \leqslant m$,

$$\mathrm{J}_n(L) = |\det S| \quad \text{where} \quad L = OS\,,$$

$S = S^*$ (S is symmetric), and $O^*O = 1$ (O is an isometry). In particular, in this case,

$$\mathrm{J}_n(L) = \sqrt{\det(L^*L)}\,.$$

Moreover, we have $\omega_n = \alpha_n = \mathcal{L}^n\big(B(0,1)\big)$.

Proof. We have $\mathrm{J}_n(L) = \mathrm{J}_n(|L'|(x))$ for all x, and the latter vanishes for $\mathcal{L}^n$ a.e. x for which $\|L\|$ is not a norm, by theorem 4.1.3. Thus, we may assume that L is injective, which implies $n \leqslant m$. Then let $L = OS$. By theorem 4.1.3 again,

$$\mathrm{J}_n(L) = \mathrm{J}_n(L) \cdot \mathcal{L}^n\big([0,1]^n\big) = \frac{\alpha_n}{\omega_n} \cdot \mathcal{H}^n\big(L([0,1]^n)\big) = \frac{\alpha_n}{\omega_n} \cdot \mathcal{H}^n\big(S[0,1]^n\big)\,.$$

The matrix S is symmetric and therefore orthogonally equivalent to some diagonal matrix $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, so

$$|\det(S)| = |\lambda_1 \cdots \lambda_n| = \mathcal{L}^n\big(D([0,1]^n)\big) = \frac{\alpha_n}{\omega_n} \cdot \mathcal{H}^n\big(D[0,1]^n\big) = \frac{\alpha_n}{\omega_n} \cdot \mathcal{H}^n\big(S([0,1]^n)\big)\,,$$

by theorem 2.4.3. Moreover, $L^*L = SO^*OS = S^2$, and

$$\mathrm{J}_n(L)^2 = \det(S)^2 = \det(L^*L)\,,$$

proving the second claim.

Now, to prove that $\mathcal{L}^n\big(B(0,1)\big) = \omega_n$, define $f : [0,\infty[\times [-\pi,\pi] \times \big[-\frac{\pi}{2}, \frac{\pi}{2}\big] \to \mathbb{R}^n$ by

$$f(r, \vartheta_1, \dots, \vartheta_{n-1}) = \big(r \cos\vartheta_{n-1} \cdots \cos\vartheta_1, r\cos\vartheta_{n-1} \cdots \cos\vartheta_2 \sin\vartheta_1, \dots, r \sin\vartheta_{n-1}\big)$$

is differentiable and an immersion at $\mathcal{L}^n$ a.e. x, and

$$\det f'(r,\vartheta) = r^{n-1} \cdot \cos^{n-2}\vartheta_{n-1} \cdots \cos\vartheta_2\,.$$

Hence, setting $Q = [0,1] \times [-\pi,\pi] \times [-\pi/2,\pi/2]^{n-2}$,

$$\omega_n = \mathcal{H}^n\big(B(0,1)\big) = \frac{\omega_n}{\alpha_n} \cdot \int_Q \mathrm{J}_n(f')\,d\mathcal{L}^n = \frac{\omega_n}{\alpha_n} \cdot \int_Q |\det f'|\,d\mathcal{L}^n\,,$$

by the above computation and the area formula 4.1.3, and this gives

$$\alpha_n = \frac{2\pi}{n} \cdot \prod_{j=1}^{n-2} \int_{-\pi/2}^{\pi/2} \cos^j(x)\,dx = \omega_n\,.$$

Thus, $\mathcal{L}^n\big(B(0,1)\big) = \alpha_n = \omega_n$. □

4.1.8. We can now give some applications. If $\gamma : \mathbb{R} \to \mathbb{R}^n$ is Lipschitz and injective (more generally: the set of double points in $\gamma(\mathbb{R})$ is $\mathcal{H}^1$ negligible), then the curve $C_t = \gamma([0,t])$ has length

$$\ell(t) = \mathcal{H}^1(C_t) = \int_0^t \|\dot\gamma\|\,d\mathcal{L}^1\,.$$

In particular, if $\dot\gamma(t) \neq 0$ for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, then ℓ is a strictly increasing function which can be inverted on $I = \ell(\mathbb{R})$. If $\|\dot\gamma\|$ is locally summable, then ℓ is continuous and I is an interval containing 0. We may then reparametrise the curve by arc length, i.e. the function defined by $\varrho(r) = \gamma(\ell^{-1}(r))$ for $r \in I$ is continuous, and $C_t = \varrho([0,\ell(t)])$.

Generalising the first example, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an injective Lipschitz map, e.g. the local chart of an embedded submanifold. Then

$$f'(x)^* f'(x) = G(x) = \big((\partial_i f(x) | \partial_j f(x))\big)_{1\leqslant i,j\leqslant n}\,,$$

and thus $g = \det G = \mathrm{J}_n(|f'|)^2$. We find the volume of $M = f(A)$, $A \subset \mathbb{R}^n$ Borel,

$$\mathcal{H}^n(M) = \int_A \sqrt{g}\,d\mathcal{L}^n\,.$$

In particular, if $m = 1$ and $f(x) = (x, h(x))$, the surface area of the graph is

$$\mathcal{H}^n\big(\mathrm{Gr}(h|A)\big) = \int_A \sqrt{1 + \|h'\|^2}\,d\mathcal{L}^n\,.$$

For the latter statement, we have used the Binet-Cauchy formula

$$\det(L^*L) = \sum_{1\leqslant k_1<\cdots<k_n\leqslant m} \det(\ell_{ik_j})_{1\leqslant i,j\leqslant n} \quad \text{for all } \ L = (\ell_{ij}) \in \mathbb{R}^{n\times m}\,.$$

The formula is valid for any n and m, both sides vanishing for $n > m$.

Remark 4.1.9. The presentation of the material in this subsection follows [AK00b], although the proofs are from [Kir94], with the necessary modifications to make them independent from the Euclidean change of variables formula.

4.2 Rectifiable Sets and Measures

In the following, let X be a metric space. Often, we shall assume that X be separable and/or complete.

Definition 4.2.1. Let $A \in \mathcal{B}(X)$. We say that A is *countably k-rectifiable* if there exist Borel sets $A_j \subset \mathbb{R}^k$ and Lipschitz functions $f_j : A_j \to X$ such that $\mathcal{H}^k\big(A \setminus \bigcup_{j=0}^{\infty} f_j(A_j)\big) = 0$.

A finite Borel measure μ on X is called *k-rectifiable* if μ is concentrated on a separable subset and $\mu = \vartheta \cdot \mathcal{H}^k \llcorner A$ for some countably k-rectifiable Borel $A \subset X$ and some Borel function $\vartheta : A \to]0, \infty[$.

Remark 4.2.2.

(i). Countably rectifiable sets are closed under countable unions.

(ii). If $X \subset Y$ where Y is metric and X is Borel in Y, then $A \in \mathcal{B}(X)$ is countably rectifiable in X if and only this is the case in Y. Indeed, Hausdorff measure on Y restricts to Hausdorff measure on X.

(iii). Clearly, If X is a Hilbert space, separable, or otherwise isometrically embedded into ℓ^∞, we may replace $f_j : A_j \to X$ in the definition by $f_j : \mathbb{R}^k \to X$ or even a single Lipschitz $f : \mathbb{R}^k \to X$, by theorem 3.1.1 and proposition 3.1.5.

(iv). By proposition 1.2.2 and theorem 1.2.5, μ is automatically concentrated on a separable subset if X is σ-compact or contains a dense subset whose cardinality is an Ulam number.

Lemma 4.2.3. Let $A \subset X$ be countably k-rectifiable. Then there exist a countable compact family $\mathcal{K} \subset \mathcal{P}(\mathbb{R}^k)$ and bi-Lipschitz maps $f_K : K \to f(K) \subset A$ such that $f_K(K)$ are pairwise disjoint and $\mathcal{H}^k\big(A \setminus \bigcup_{K \in \mathcal{K}} f_K(K)\big) = 0$.

Proof. Let $A_j \subset \mathbb{R}^k$ be Borel, $f_j : A_j \to X$ be Lipschitz, such that

$$\mathcal{H}^k(A \setminus \bigcup_{j=0}^{\infty} f(A_j)) = 0 .$$

Let B_j be the set of $x \in A_j$ such that $|f_j'|(x)$ exists and is not a norm. Then

$$\mathcal{H}^k(f_j(B_j)) \leqslant \int N(f_j|B_j, \sqcup)\, d\mathcal{H}^k = \int \mathrm{J}_n(|f_j'|)\, d\mathcal{L}^k = 0 ,$$

by the area formula 4.1.3. Thus, shrinking A_j, we may assume that $|f_j'|(x)$ is a norm whenever it exists. By proposition 3.3.11, by further subdividing the A_j, we may assume that the f_j are bi-Lipschitz onto their images.

Now, define Borel sets C_j by

$$C_j = A_j \setminus f_j^{-1}\Big(A \cap \bigcup_{i<j} f_i(A_i)\Big) \quad \text{for all } j \in \mathbb{N} .$$

Then $f_j(C_j) \subset A$ are pairwise disjoint and $\mathcal{H}^k$ almost cover A. Almost covering C_j by countably many disjoint compacts (theorem 1.1.9 (iii)) changes the image $f_j(C_j)$ only by a zero set (theorem 2.2.2). □

Definition 4.2.4. Let E be a separable Banach space, $Y = E^*$, $S \in \mathcal{B}(Y)$, $S = f(B)$ for some $B \in \mathcal{B}(\mathbb{R}^k)$ and a Lipschitz map $f : \mathbb{R}^k \to Y$ such that $f|B$ is injective. Then, for any $y = f(x) \in S$ such that f is metrically and weak* differentiable at x, with $\mathrm{J}_k(f'_\sigma(x)) > 0$, define the *approximate tangent space* of S at x by

$$\mathrm{Tan}^k(S, y) = f'_\sigma(\mathbb{R}^k) .$$

If S is countably $\mathcal{H}^k$-rectifiable, $\mathcal{H}^k\big(S \setminus \bigcup_i f_i(B_i)\big) = 0$, where $f_i : B_i \to Y$ are bi-Lipschitz onto their images, then define the approximate tangent space by

$$\mathrm{Tan}^k(S, y) = \mathrm{Tan}^k(f_i(B_i), y) \quad \text{for all } \ y \in S \cap f_i(B_i) .$$

We need to see that this definition is well-posed, and moreover, a.e. independent of the choice of parametrisations.

Lemma 4.2.5. Let $S_j = f_j(B_j) \in \mathcal{B}(Y)$, $f_j \in \mathrm{Lip}(\mathbb{R}^k, Y)$, such that $f_j|B_j$ are injective, $j = 1, 2$. Then

$$\mathrm{Tan}^k(S_1, y) = \mathrm{Tan}^k(S_2, y) \quad \text{for } \mathcal{H}^k \text{ a.e.} \quad y \in S_1 \cap S_2 .$$

The conclusion holds also for any pair of countably k-rectifiable sets S_j, $j = 1, 2$.

Proof. Let $K \subset S_1 \cap S_2$ be closed. We prove $\subset$ for $\mathcal{H}^K$ a.e. $y \in K$. Then the statement follows by symmetry and inner regularity.

Thus, set $K_j = B_j \cap f_j^{-1}(K)$. Moreover, let K'_j be the subset of $\mathcal{L}^k$ density points for K_j, at which f_j is metrically and weak* differentiable, with non-vanishing Jacobian. We will prove $\subset$ at any $y \in K' = f_1(K'_1) \cap f_2(K'_2)$.

Let $y = f_1(u) = f_2(v) \in K'$. Since u is an $\mathcal{L}^k$ density point for K_1, there is an orthonormal basis $e_1, \ldots, e_k$ of $\mathbb{R}^k$ and a sequence $t_m \to 0+$ such that $u + t_m e_i \in K_1$ for all $i = 1, \ldots, k$, $m \in \mathbb{N}$. Fix $1 \leqslant i \leqslant k$, and let $u_m = u + t_m e_i \to u$. Then $y_m = f_1(u_m) \to y$, and we may assume (possibly passing to a subsequence) that for $v_m = f_2^{-1}(y_m)$, the sequence $\|v - v_m\|^{-1} \cdot (v - v_m)$ converges to some $e \in \mathbb{S}^{k-1}$. (Note that $u_m \neq u$ implies $y_m \neq y$, which implies $v_m \neq v$.)

Now, in the $\sigma(Y, E)$ topology,

$$\begin{aligned} f'_{1\sigma}(u)e_i &= \lim_{m\to\infty} \frac{y_m - y}{t_m} \\ &= \lim_{m\to\infty} \frac{\|y_m - y\|}{t_m} \cdot \frac{\|v_m - v\|}{\|f_2(v_m) - f_2(v)\|} \cdot \frac{f_2(v_m) - f_2(v)}{\|v_m - v\|} = \frac{|f'_{1\sigma}|(u)e_i}{|f'_{2\sigma}(v)e|} \cdot f'_{2\sigma}(v)e , \end{aligned}$$

so $f'_{1\sigma}(y)e_i \in \mathrm{Tan}^k(S_2, y)$. Since i was arbitrary, the assertion follows. □

Similar arguments give the following more intrinsic characterisation of the approximate tangent space (by secant vectors).

Proposition 4.2.6. Let $S \in \mathcal{B}(Y)$ be countably k-rectifiable. Then there is a countable Borel $\mathcal{H}^k$ almost cover (S_k) of S such that for all k,

$$\mathrm{Tan}^k(S_k, y) \cap \mathbb{S}(Y) = \left\{ v \in Y \;\middle|\; \exists u_m \in S_k \,:\, v = \lim_m \frac{u_m - y}{\|u_m - y\|} \text{ in } Y_\sigma \right\}$$

for $\mathcal{H}^k$ a.e. $y \in S_k$, where $\mathbb{S}(Y)$ is the unit sphere of $Y = E^*$.

The following proposition enables us to define the approximate tangent space independent of a particular embedding into a dual Banach space (see below).

Proposition 4.2.7. Let $S \subset Y = E^*$ be countably k-rectifiable, and $\mathcal{H}^k(S) < \infty$. For $\mathcal{H}^k$ a.e. $y \in S$, there exist a Borel $S_y \subset Y$, such that $\Theta^*(S \setminus S_y, y) = 0$, and a weakly continuous map $\pi_y : Y \to \mathrm{Tan}^k(S, y)$ such that $\pi_y | \mathrm{Tan}^k(S, y) = \mathrm{id}$, and

$$\lim_{r \to 0+} \sup \left\{ \left| \frac{\|\pi_y(u - v)\|}{\|u - v\|} - 1 \right| \;\middle|\; u \neq v\,,\, u, v \in B(y, r) \cap S_y \right\} = 0\,.$$

Proof. W.l.o.g., let $S \subset f(\mathbb{R}^k)$ for some Lipschitz map $f : \mathbb{R}^k \to Y$. Let a countable Borel partition (B_i) of the set of points in $\mathbb{R}^k$ be given at which f is weak* and metrically differentiable, and the Jacobian is non-vanishing, such that $f|B_i$ is bi-Lipschitz onto $f_i(B_i)$ for all i (proposition 3.3.11). Let (K_j) be a countable family of compacts, such that

$$\big| \|f(u) - f(v)\| - |f'|(v)(u - v) \big| \leqslant \omega_j(\|u - v\|) \cdot \|u - v\| \quad \text{for all } \; u \in \mathbb{R}^k\,,\, v \in K_j\,,$$

where ω_j are moduli of continuity for $|f'||K_j$, by the metric mean value inequality (theorem 3.3.9). Let $S_{ij} = f(B_i \cap K_j)$. By the area formula (theorem 4.1.3), the set of all $y = f(x)$ such that $\mathrm{J}_k(|f'|(x)) = 0$ is $\mathcal{H}^k$ negligible, so (S_{ij}) $\mathcal{H}^k$ almost covers S.

Let $y = f(x) \in S_{ij}$. Then $f'_\sigma(x)$ is injective, and $\mathrm{Tan}^k(S, y)$ is k-dimensional. Then the weakly continuous projection π_y is obtained by choosing a basis $u_1, \cdots, u_k$ of $\mathrm{Tan}^k(S, y)$ and the dual basis $\langle u_m : \nu_n \rangle = \delta_{mn}$. There exist weakly continuous extensions $\mu_n \in (Y_\sigma)^* = E$ of ν_n by the Hahn-Banach theorem. Then $\pi_y(u) = \sum_{m=1}^k \langle u : \mu_n \rangle \cdot u_m$ gives the desired map.

By corollary 2.3.5, for any fixed i, j, $\mathcal{H}^k$ a.e. $z \in S_{ij}$ satisfies $\Theta_k^*(S \setminus S_{ij}, z) = 0$. Then the assertion will follow for $S_y = S_{ij}$ as soon as we have established the convergence statement for each of these sets.

Since $f_i|B_i$ is bi-Lipschitz, the convergence will follow from

$$\lim_{r \to 0+} \sup \left\{ \left| \frac{|f'|(x)(u - v)}{\|f(u) - f(v)\|} - 1 \right| \;\middle|\; u \neq v\,,\, u, v \in B(x, r) \cap B_i \cap K_j \right\} = 0$$

and

$$\lim_{r\to 0+}\sup\left\{\left|\frac{\|f'_\sigma(x)(u-v)\|}{\|\pi_y\big(f(u)-f(v)\big)\|}-1\right| \;\middle|\; u\neq v\,,\, u,v\in B(x,r)\cap B_i\cap K_j\right\}=0$$

The first of these two statements follows from

$$\left|\frac{|f'|(x)(u-v)}{\|f(u)-f(v)\|}-1\right|\leqslant \omega_j(\|u-v\|)\cdot\frac{\|u-v\|}{\|f(u)-f(v)\|}\quad\text{for all } u\neq v\,,\, u,v\in B_i\cap K_j\,,$$

since $f|B_i$ is bi-Lipschitz. Similarly, for the second,

$$\left|\frac{\|f'_\sigma(x)(u-v)\|}{\|\pi_y\big(f(u)-f(v)\big)\|}-1\right|\leqslant\frac{\|\pi_y\big(f'_\sigma(x)(u-v)-f(u)+f(v)\big)\|}{\|\pi_y\big(f(u)-f(v)\big)\|}\,. \tag{$*$}$$

Now, $\|\pi_y(\sqcup)\|$ is weakly l.s.c., so

$$\lim_{r\to 0+}\sup_{u\neq v\,,\,u,v\in B(x,r)}\frac{\|\pi_y\big(f'_\sigma(x)(u-v)-f(u)+f(v)\big)\|}{\|u-v\|}=0\,.$$

Moreover, for any sequence $(u_m,v_m)\to(x,x)$, $u_m\neq v_m$, $u_m,v_m\in B_i\cap K_j$, such that $\frac{u_m-v_m}{\|u_m-v_m\|}$ converges to some $e\in\mathbb{S}^{m-1}$, we find

$$\lim_{m\to\infty}\frac{f(u_m)-f(v_m)}{\|u_m-v_m\|}=f'_\sigma(x)e\quad\text{in}\quad Y_\sigma\,.$$

Thus, again using the weak* lower semicontinuity of $\|\pi_y(\sqcup)\|$,

$$\begin{aligned}\liminf_{r\to 0+}\inf_{u\neq v\,,\,u,v\in B(x,r)}\frac{\|\pi_y\big(f(u)-f(v)\big)\|}{\|u-v\|}&\geqslant\inf_{e\in\mathbb{S}^{k-1}}\|\pi_y(f'_\sigma(x)e)\|\\&=\inf_{e\in\mathbb{S}^{k-1}}\|f'_\sigma(x)e\|>0\,.\end{aligned}$$

Hence, the right hand side of $(*)$ is the product of a bounded term and a term converging to zero, so the assertion follows. ☐

Definition 4.2.8. Given a metric space X and a countably k-rectifiable $S\in\mathcal{B}(X)$, fix an isometric embedding $j:S\to Y=E^*$, E a separable Banach space. Define

$$\operatorname{Tan}^k(S,x)=\operatorname{Tan}^k(j(S),j(x))\quad\text{for all } x\in S \text{ for which this make sense.}$$

Then $\operatorname{Tan}^k(S,x)$ is well-defined up to isometries $\mathcal{H}^k$ a.e., by propositions 4.2.6 and 4.2.7. We round off the section with characterisations of rectifiability for sets and measures.

Definition 4.2.9. Let E,F be Banach spaces. Consider the *weak Grassmannian variety*

$$\Pi_k(E^*,F)=\big\{L:E^*\to Y\;\big|\;L\text{ linear,weak}^*\text{ continuous, }\operatorname{rk}L=k\big\}\,.$$

Define a pseudometric γ on $\Pi_k(E^*, F)$ by

$$\gamma(L, L') = \sup\{|L(x) - L'(x)| \mid x \in E^*, \|x\| \leqslant 1\}.$$

Here, recall that a *pseudometric* is a function satisfying all the axioms of a metric save the separation axiom. Of course, a pseudometric induces a (usually non-Hausdorff) topology in the same way as does a metric. The following lemma shall be useful.

Lemma 4.2.10. If E is separable, then so is $\Pi_k(E^*, F)$, in the topology induced by γ.

Proof. Any $L \in \Pi_k(E^*, F)$ may be factored through $\mathbb{R}^k$. The proof uses the separability of E and of the set of norms (i.e. closed convex symmetric neighbourhoods of 0) in $\mathbb{R}^k$, cf. 3.3.10. For details, we refer to [AK00b, lem. 6.1]. □

Proposition 4.2.11. Let E be a separable Banach space, and $S \subset Y = E^*$ a separable subset. If, for each $x \in S$, there are $\varepsilon_x, r_x > 0$, and $\pi_x \in \Pi_k(Y, Y)$ such that

$$\|\pi_x(y - x)\| \geqslant \varepsilon_x \|y - x\| \quad \text{for all } y \in B(x, r_x) \cap S,$$

then there exist Lipschitz functions $f_m : \mathbb{R}^k \to Y$, such that $S \subset \bigcup_{m=0}^{\infty} f_m(\mathbb{R}^k)$. In particular, if S is Borel, then it is countably k-rectifiable.

Proof. Let $S_n = \{x \in S \mid \min(\varepsilon_x, r_x) \geqslant \frac{1}{n}\}$. Then $S = \bigcup_{n=0}^{\infty} S_n$. Select a dense sequence $(\pi_m) \subset \Pi_k(Y, Y)$, by lemma 4.2.10. Define

$$S_{nm} = \left\{x \in S_n \;\middle|\; \gamma(\pi_x, \pi_m) < \frac{1}{2n}\right\} \quad \text{and} \quad V_m = \pi_m(Y).$$

Note that V_m spans a k-dimensional subspace of Y, and is thus contained in the image of a linear (and hence Lipschitz) function $\mathbb{R}^k \to Y$.

Now, let $S_{nm} = \bigcup_{\ell=0}^{\infty} S_{nm\ell}$ where $\operatorname{diam} S_{nm\ell} < \frac{1}{n}$ (S is separable). If $u, v \in S_{nm\ell}$, then

$$\|\pi_m(u - v)\| \geqslant \|\pi_v(u - v)\| - \frac{1}{2n} \cdot \|u - v\| \geqslant \frac{1}{2n} \cdot \|u - v\|,$$

so $\pi_m : S_{nm\ell} \to V_m$ is bi-Lipschitz onto its image. By proposition 3.1.5, there exists a Lipschitz function $f_{nm\ell} : \mathbb{R}^k \to Y$ containing $S_{nm\ell}$ in its image. □

Proposition 4.2.12. Let μ be a finite Borel measure on $Y = E^*$, E separable. Then μ is k-rectifiable if and only if for μ a.e. $x \in X$,

(i). we have

$$0 < \Theta_{k*}(\mu, x) \leqslant \Theta_k^*(\mu, x) < \infty, \quad \text{and}$$

(ii). there exist $\varepsilon_x > 0$ and $\pi_x \in \Pi_k(Y, Y)$ such that for

$$C_x = \{y \in Y \mid \|\pi_x(y - x)\| \leqslant \varepsilon_x \cdot \|x - y\|\},$$

we have $\Theta_k(\mu \llcorner C_x, x) = 0$.

Proof. Because Y has no isolated points, the lower density condition shows that μ is concentrated on a Borel set S σ-finite w.r.t. $\mathcal{H}^k$, by theorem 2.3.3. Similarly, the upper density condition shows that μ is absolutely continuous w.r.t. $\mathcal{H}^k \llcorner S$. Then the assertion follows from theorem 1.9.5 as soon as we can prove that S is countably k-rectifiable.

To that end, it suffices to verify the assumptions of proposition 4.2.11 the sets

$$S_\delta = \left\{ x \in S \;\middle|\; \forall 0 < r \leqslant \delta : \frac{\mu\big(B(x,r)\big)}{r^k} \geqslant \delta \right\}$$

defined for $\delta > 0$. To that end, fix $x \in S_\delta$, and $\gamma \in]0,1[$, so that

$$\frac{\varepsilon_x}{2} + \gamma \cdot \|\pi_x\| \leqslant \varepsilon_x \cdot (1-\gamma)\,.$$

We claim that for y sufficiently close to x, we have

$$2 \cdot \big\|\pi_x\big(y-x\big)\big\| \geqslant \varepsilon_x \cdot \|y-x\|\,. \tag{$*$}$$

Indeed, for any $z \in B(y, \gamma \cdot \|y-x\|)$, we have

$$\|y-x\| \leqslant \|y-z\| + \|z-x\| \leqslant \gamma \cdot \|y-x\| + \|z-x\|\,,$$

so $\|y-x\| \leqslant \frac{1}{1-\gamma} \cdot \|z-x\|$. Let $r = \|y-x\|$. Whenever $(*)$ fails, we find

$$\begin{aligned} \|\pi_x(z-x)\| &\leqslant \|\pi_x(y-x)\| + \|\pi_x(z-y)\| \\ &\leqslant \frac{\varepsilon_x r}{2} + \|\pi_x\| \cdot \|z-y\| \leqslant \left(\frac{\varepsilon_x}{2} + \gamma\|\pi_x\|\right) \cdot r \leqslant \varepsilon_x \cdot \|z-x\|\,. \end{aligned}$$

Hence, $B(y, \gamma r) \subset C_x$. Thus, if $(*)$ fails for arbitrarily small r, we find

$$\delta\gamma^k \leqslant \frac{\mu\big(B(x,\gamma r)\big)}{r^k} \leqslant \frac{\mu\big(B(x,r)\big)}{r^k} \to 0\,,$$

a contradiction! Hence, $(*)$ holds for y close to x, the required condition. □

4.3 Tangential differential and general area formula

The next natural step in our attempt to reconstruct the elements of differential geometry in a purely measurable metric setup is to remove the explicit reference to Lipschitz parametrisations.

Theorem 4.3.1. Let $Y = E^*$ and $Z = F^*$ be duals of seperable Banach spaces, $S \subset Y$ countably k-rectifiable, $g : S \to Z$ Lipschitz, and $\mu = \vartheta \cdot \mathcal{H}^k \llcorner S$, where $\vartheta : S \to]0,\infty[$ is an $\mathcal{H}^k$ summable function. Then, for $\mathcal{H}^k$ a.e. $x \in S$, there exist a continuous linear

$L_x : Y_\sigma \to Z_\sigma$ and a Borel $S^x \subset S$ such that

$$\Theta^*_k(\mu \llcorner S_x, x) = 0 \quad \text{and} \quad \lim_{S\setminus S^x \ni y \to x} \frac{d_\sigma\big(g(y), g(x) + L_x(y-x)\big)}{\|y-x\|} = 0\,,$$

d_σ denoting the canonical metric defining the $\sigma(Z,F)$ topology on the unit ball of Z, cf. 3.1.7.

Moreover, L_x is uniquely determined on $\mathrm{Tan}^k(S,x)$. Writing $d^S g(x) = L_x|\mathrm{Tan}^k(S,x)$, this *tangential differential* is characterised by

$$(g \circ h)'_\sigma(a) = d^S g\big(h(a)\big) \circ h'_\sigma(a) \quad \text{for } \mathcal{L}^k \text{ a.e.} \quad a \in A\,,$$

and every Lipschitz $h : A \to S$, where $A \subset \mathbb{R}^k$.

Proof. Up to $\mathcal{H}^k$ negligible sets, we write $S = \bigcup_{i=0}^\infty f(B_i)$ where $f : \mathbb{R}^k \to Y$ is Lipschitz, $f|B_i$ is bi-Lipschitz with non-vanishing Jacobian, and metrically and weak* differentiable. Also assume that $g \circ f$ is weak* differentiable on each of the B_i. Fix any $y = f(x) \in f(B_i)$, and define L_y on $\mathrm{Tan}^k(f(B_i), f(x))$ by

$$L_y\big(f'_\sigma(x)u\big) = (g \circ f)'_\sigma(x)u \quad \text{for all} \;\; u \in \mathbb{R}^k\,,$$

and applying the projection π_y from proposition 4.2.7. The statement follows from the non-trivial density result [Kir94, th. 5.4.]. □

Definition 4.3.2. Let X, Y be separable metric spaces, $S \subset X$ countably k-rectifiable, and $g : S \to Y$ Lipschitz. Choose isometries $j_X : X \to E^*$ and $j_Y : Y \to F^*$ where E, F are separable Banach spaces. Then define *Jacobian* of g by

$$\mathrm{J}_k(d^S g)(x) = \mathrm{J}_k(d^{j_X(S)}(j_Y \circ g \circ j_X^{-1})(j_X(x))) \quad \text{for all} \;\; x \in S$$

where this makes sense. That this is $\mathcal{H}^k$ a.e. well-defined is part of the following theorem.

General Change of Variables Formula 4.3.3. Let $g : X \to Y$ be Lipschitz, where X and Y are separable, and $S \subset X$ be countably k-rectifiable.

(i). The Jacobian $\mathrm{J}_k(d^S g)$ is $\mathcal{H}^k$ a.e. well-defined.

(ii). For all Borel maps $\vartheta : S \to [0,\infty]$, we have

$$\int_S \vartheta \cdot \mathrm{J}_k(d^S g)\, d\mathcal{H}^k = \int_Y \sum_{x \in S \cap g^{-1}(y)} \vartheta(x)\, d\mathcal{H}^k(y)\,.$$

(iii). For all $A \in \mathcal{B}(X)$ and all Borel functions $\vartheta : Y \to [0,\infty]$, we have

$$\int_A \vartheta(g(x)) \cdot \mathrm{J}_k(d^S g)(x)\, d\mathcal{H}^k(x) = \int_Y \vartheta(y) N(g|A, y)\, d\mathcal{H}^k(y)\,.$$

Proof. Claim (ii) follows from (iii) in the usual way. Moreover, the right hand side in (iii) is independent of embeddings, so (i) also follows from (iii) one this has been established for an arbitrary embedding. By σ-additivity, we may assume $S \subset f(\mathbb{R}^k)$ where f is bi-Lipschitz and has non-vanishing Jacobian. Then the defining equation for the tangential differential, the chain rule for the Jacobian, and the change of variables formula (corollary 4.1.6) give

$$\begin{aligned}\int_A \vartheta(g(x))\mathrm{J}_k(d^S g)(x)\,d\mathcal{H}^k(x) &= \int_A \vartheta(g(x))\mathrm{J}_k((g\circ f)'_\sigma(f^{-1}(x)))\mathrm{J}_k(f'_\sigma(f^{-1}(x)))^{-1}\,d\mathcal{H}^k(x)\\ &= \int_{f^{-1}(A)} \vartheta(g(f(x)))\mathrm{J}_k((g\circ f)'_\sigma(x))\,d\mathcal{L}^k(x)\\ &= \int_Y \vartheta(y)N(g\circ f|f^{-1}(A),y)\,d\mathcal{H}^k(y)\,,\end{aligned}$$

whence the assertion, since $N(g|A,y) = N(g\circ f|f^{-1}(A),y)$, f being injective. — □

Remark 4.3.4. The main body of this section follows [AK00b], with some aspects derived from [Kir94], and lemma 4.2.3 being [AK00a, lem. 4.5].

5 Currents

5.1 Metric Functionals

In the following, let (X, d) be a metric space; usually, we shall assume X to be complete.

Definition 5.1.1. Denote by $\mathrm{Lip}(X)$ the set of all Lipschitz functions $X \to \mathbb{R}$, and by $\mathrm{Lip}_b(X)$ the set of all bounded Lipschitz maps. Let $\mathcal{D}^k(X) = \mathrm{Lip}_b(X) \times \mathrm{Lip}(X)^k$ for all $k \in \mathbb{N}$, and $\mathrm{MF}_k(X)$ be the set of maps $T : \mathcal{D}^k(X) \to \mathbb{R}$ such that $|T|$ is positively homogeneous and subadditive in each variable. The elements of $\mathrm{MF}_k(X)$ are called *k-dimensional metric functionals*.

Elements $\omega = (f, \pi_1, \dots, \pi_k) \in \mathcal{D}^k(X)$ shall be denoted $\omega = f\,d\pi_1 \wedge \cdots \wedge d\pi_k$ or even $f\,d\pi$ where $\pi = (\pi_1, \dots, \pi_k)$. In the sequel, it shall be become clear why this notation is justified.

The *exterior differential* is the degree 1 linear endomorphism d of the graded vector space $\mathcal{D}^\bullet(X) = \sum_{k=0}^{\oplus\infty} \mathcal{D}^k(X)$, defined by

$$d\big(f\,d\pi_1 \wedge \cdots d\pi_k\big) = df \wedge d\pi_1 \wedge \cdots \wedge d\pi_k\,.$$

The *boundary operator* is the dual degree -1 linear endomorphism of the graded vector space $\mathrm{MF}_\bullet(X) = \sum_{k=0}^{\oplus\infty} \mathrm{MF}_k(X)$, defined by

$$\langle \omega : \partial T \rangle = \langle d\omega : T \rangle \quad \text{for all } \omega \in \mathcal{D}^k(X)\,,\ T \in \mathrm{MF}_{k+1}(X)\,,\ k \geqslant 0\,.$$

Assume $\varphi : X \to Y$ is Lipschitz. Then the *pullback* along φ is the degree 0 linear map $\varphi^\bullet : \mathcal{D}^\bullet(Y) \to \mathcal{D}^\bullet(X)$ defined by

$$\varphi^\bullet\big(f\,d\pi_1 \wedge \cdots \wedge d\pi_k\big) = (f \circ \varphi)\,d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_k \circ \varphi)\,.$$

Dually, we obtain the *pushforward* $\varphi_\bullet : \mathrm{MF}_\bullet(X) \to \mathrm{MF}_\bullet(Y)$, the degree 0 linear map given by

$$\langle \omega : \varphi_\bullet T \rangle = \langle \varphi^\bullet \omega : T \rangle \quad \text{for all } \omega \in \mathcal{D}^k(Y)\,,\ T \in \mathrm{MF}_k(X)\,.$$

Let $T \in \mathrm{MF}_k(X)$ and $\omega = g\,d\tau_1 \wedge \cdots \wedge d\tau_m \in \mathcal{D}^m(X)$ where $0 \leqslant m \leqslant k$. Then we define the *contraction* $T \llcorner \omega \in \mathrm{MF}_{k-m}(X)$ by

$$\langle f\,d\pi_1 \wedge \cdots \wedge d\pi_{k-m} : T \llcorner \omega \rangle = \langle fg\,d\tau_1 \wedge \cdots \wedge d\tau_m \wedge d\pi_1 \wedge \cdots \wedge d\pi_{k-m} : T \rangle\,.$$

We say that $T \in \mathrm{MF}_k(X)$ has *finite mass* if for some finite Borel measure μ on X,

$$\big|\langle f\,d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle\big| \leqslant \int |f|\,d\mu \cdot \prod_{j=1}^k \mathrm{Lip}(\pi_j) \quad \text{for all } f\,d\pi_1 \wedge \cdots \wedge d\pi_k \in \mathcal{D}^k(X)\,.$$

Because $\mathrm{Lip}_b(X)$ is dense in $\mathbf{L}^1(\mu)$, T can be uniquely continuously extended to the space $\mathcal{B}_b(X) \times \mathrm{Lip}(X)^k$, where $\mathcal{B}_b(X)$ are the bounded Borel functions on X.

Proposition 5.1.2. Let $T \in \mathrm{MF}_k(X)$. If T has finite mass, then there exists a smallest finite Borel measure μ on X satisfying

$$\big|\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle\big| \leqslant \int |f|\, d\mu \cdot \prod_{j=1}^{k} \mathrm{Lip}(\pi_j) \quad \text{for all } f\, d\pi_1 \wedge \cdots \wedge d\pi_k \in \mathcal{D}^k(X)\,, \tag{$*$}$$

This measure is called the *mass of* T, denoted $\|T\|$.

In fact, T has finite mass if and only both of the following conditions hold:

(i). There exists a constant $0 \leqslant M < \infty$, so that for $(f_i) \subset \mathrm{Lip}_b(X)$, $(\pi^i_j) \subset \mathrm{Lip}(X)$,

$$\sum_{i=0}^{\infty} \big|\langle f_i\, d\pi^i_1 \wedge \cdots \wedge d\pi^i_n : T\rangle\big| \leqslant M \quad \text{whenever} \quad \textstyle\sum_i |f_i| \leqslant 1\,,\ \mathrm{Lip}(\pi^i_j) \leqslant 1\,;$$

(ii). for any $(\pi_j) \subset \mathrm{Lip}(X)$, the sublinear functional $|T(\sqcup, \pi_1, \ldots, \pi_n)|$ on $\mathrm{Lip}_b(X)$ is continuous on uniformly bounded increasing sequences, i.e., sequences $(f_k) \subset \mathrm{Lip}_b(X)$ is such that (f_k) is pointwise increasing and $\sup_{x\in X, k\in\mathbb{N}} |f_k(x)| < \infty$.

Moreover, $\|T\|$ is given by

$$\|T\|(B) = \sup\Big\{ \sum_{j=0}^{\infty} \big|\langle 1_{B_j}\, d\pi^i_1 \wedge \cdots \wedge d\pi^i_k : T\rangle\big| \ \Big|\ B = \biguplus_{j=0}^{\infty} B_j\,,\ \pi^i_j \in \mathrm{Lip}_1(X) \Big\}\,, \tag{$**$}$$

for all $B \in \mathcal{B}(X)$, and $\|T\|(M)$ is the least constant in (i).

Proof. Let conditions (i) and (ii) hold. Let $\pi = (\pi_1, \ldots, \pi_k) \in \mathrm{Lip}_1(X)^k$. For $U \subset X$ open, define

$$\mu_\pi(U) = \sup\big\{ |\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| \ \big|\ f \in \mathrm{Lip}_b(X)\,,\ |f| \leqslant 1_U \big\} \text{ where } \sup \varnothing = 0\,.$$

Let $U \subset \bigcup_{j=0}^{\infty} U_j$ where $U_j \subset X$ are open. We claim $\mu_\pi(U) \leqslant \sum_{j=0}^{\infty} \mu_\pi(U_j)$. To that end, let $\psi^N_j = \min\big(1, N \cdot \mathrm{dist}(\sqcup, X \setminus U_j)\big)$,

$$\varphi^N_j = \frac{\psi^N_j}{\frac{1}{N} + \sum_{j=0}^{N} \psi^N_j}\,,\ \text{and } \phi_N = \sum_{j=0}^{N} \varphi^N_j\,,\ \text{so } \phi_N = \frac{1}{1 + \big(N \cdot \sum_{j=0}^{N} \psi^N_j\big)^{-1}} \ \text{on } \bigcup_{j=0}^{\infty} U_j\,.$$

Then $0 \leqslant \phi_N \leqslant \phi_{N+1} \leqslant 1$, and $1 = \lim_N \phi_N$ point-wise on $\bigcup_{j=0}^{\infty} U_j$.

Indeed, if $\mathrm{dist}(x, X \setminus U_j) < \frac{1}{N}$, then

$$\psi^N_j(x) = N \cdot \mathrm{dist}(x, X \setminus U_j) \leqslant (N+1) \cdot \mathrm{dist}(x, X \setminus U_j) \leqslant \psi^{N+1}_j(x)\,,$$

and if $\mathrm{dist}(x, X \setminus U_j) \geqslant \frac{1}{N}$, then $\mathrm{dist}(x, X \setminus U_j) > \frac{1}{N+1}$, and $\psi_j^N(x) = 1 = \psi_j^{N+1}(x)$. Thus, $\phi_N \leqslant \phi_{N+1}$. If $x \in \bigcup_{j=0}^{\infty} U_j$, then $x \in U_j$ for some $j \in \mathbb{N}$, so

$$\phi_N(x) \geqslant \frac{N}{N+1} \quad \text{for all } x\,,\ N > \mathrm{dist}(x, X \setminus U_j)\,,$$

which proves that $\lim_N \phi_N(x) = 1$.

For $f \in \mathrm{Lip}_b(X)$, $|f| \leqslant 1_U$, condition (ii) implies

$$|\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| = \lim_N \Big|\sum_{j=0}^{N} \langle f \cdot \varphi_j^N\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle\Big| \leqslant \sum_{j=0}^{\infty} \mu_\pi(U_j)\,,$$

proving our claim. Now, we may extend μ_π to arbitrary sets by

$$\mu_\pi(A) = \inf\Big\{\textstyle\sum_{U\in\mathcal{U}} \mu_\pi(U) \;\Big|\; A \subset \bigcup \mathcal{U}\,,\ \mathcal{U} \text{ countable open family}\Big\}\,.$$

As in the proof of proposition 2.1.2, we see that μ_π is a Borel measure. Because of condition (i), it is finite. Next, we claim that

$$|\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| \leqslant \int |f|\, d\mu_\pi \quad \text{for all } f \in \mathrm{Lip}_b(X)\,.$$

Because of $f = f^+ - f^-$, $|f| = f^+ + f^-$, and the subadditivity of $|T|$, we may assume $f \geqslant 0$. Set $f_t = \min(t, f)$ for $t \in \mathbb{R}$. Then, for all $s > t$,

$$\begin{aligned}\big||\langle f_s\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| - |\langle f_t\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle|\big| &\leqslant |\langle (f_s - f_t)\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| \\ &\leqslant (s-t)\cdot \mu_\pi\{f > t\}\,,\end{aligned}$$

since $f_s - f_t = f - f = 0$ on $\{f \leqslant t\}$, and $f_s - f_t = f_s - t \leqslant s - t$ on $\{f > t\}$. Hence, $g(t) = |\langle f_t\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle|$ is a Lipschitz function, and $|g'(t)| \leqslant \phi(t) = \mu_\pi\{f > t\}$ whenever g is differentiable and ϕ is continuous (which is $\mathcal{L}^1$ a.e., since ϕ is decreasing). Thus,

$$\begin{aligned}|\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| &= \int_0^\infty g'(t)\, dt \leqslant \int_0^\infty \mu_\pi\{f > t\}\, dt \\ &= \int \int_0^\infty 1_{\{f>t\}}(x)\, dt\, d\mu_\pi(x) = \int f(x)\, d\mu_\pi(x)\,,\end{aligned}$$

proving our claim.

Now, $\mu(A) = \sup_{\pi\in\mathrm{Lip}_1(X)} \mu_\pi(A)$ for all $A \in \mathcal{P}(X)$ defines a finite Borel measure satisfying $(*)$. If ν is any finite Borel measure satisfying $(*)$, then the set function τ on $\mathcal{B}(X)$ defined by the right hand side of $(**)$ clearly satisfies $\tau \leqslant \nu$. On the other hand,

$$\mu_\pi(B) \leqslant |\langle 1_B\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle| \leqslant \tau(B) \quad \text{for all } B \in \mathcal{B}(X)\,,\ \pi \in \mathrm{Lip}_1(X)^k\,.$$

This implies that $\mu \leqslant \nu$, by the Borel regularity of the finite Borel measures μ and ν. Hence, μ is the smallest finite Borel measure satisfying $(*)$. Also, since μ satisfies $(*)$, we find $\mu = \tau$ on $\mathcal{B}(X)$.

The statement about least constants follows from the equality

$$\mu(X) = \sup\Big\{ \sum_{i=0}^{\infty} \mu_{\pi^i}(f_i) \;\Big|\; f_i \in \mathrm{Lip}_b(X)\,,\ \sum_{i=0}^{\infty} |f_i| \leqslant 1\,,\ \pi^i_j \in \mathrm{Lip}_1(X) \Big\}\,,$$

which is immediate from the definition of μ. □

5.1.3. Metric functionals of finite mass behave well under push-forward and contraction. Indeed, from $(*)$, we immediately have

$$\|\varphi_\bullet T\| \leqslant \mathrm{Lip}(\varphi)^k \cdot \varphi_\bullet \|T\| \quad \text{for all } \varphi \in \mathrm{Lip}(X, Y)\,,$$

where the right hand side is the image measure. In particular, if φ is an isometry, we have equality. The defining identity for the push-forward remains valid for $f\,d\pi_1 \wedge \cdots d\pi_k$ where $f \in \mathcal{B}_b(X)$.

As to contraction, it makes sense for $\omega = f\,d\pi_1 \wedge \cdots \wedge d\pi_k$ where $f \in \mathcal{B}_b(X)$, and

$$\|T \llcorner \omega\| \leqslant \|f\|_\infty \cdot \prod_{j=1}^{k} \mathrm{Lip}(\pi_j) \cdot \|T\|\,,$$

as follows from $(*)$. If $\omega = 1_C$, we write $T \llcorner C$ for $T \llcorner \omega$.

5.2 Currents

Definition 5.2.1. Let $k \in \mathbb{N}$. A *k-dimensional current* on X is a k-dimensional metric functional of finite mass $T \in \mathrm{MF}_k(X)$ satisfying the following additional hypotheses:

(i). T is $(k+1)$-linear, and $\langle f\,d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle = 0$ whenever π_i is constant on a neighbourhood of $\{f \neq 0\}$ for some i, and

(ii). Whenever $\pi_j = \lim_i \pi^i_j$ point-wise and $\mathrm{Lip}(\pi^i_j) \leqslant C$ for some $C > 0$, then

$$\langle f\,d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle = \lim_i \langle f\,d\pi^i_1 \wedge \cdots \wedge d\pi^i_k : T \rangle\,.$$

The *support* of T is $\operatorname{supp} T = \operatorname{supp}\|T\|$. The vector space of k-dimensional currents is denoted $\mathbf{M}_k(X)$. It is a normed space via the norm $\mathbf{M}(T) = \|T\|(X)$.

Proposition 5.2.2. The space $\mathbf{M}_k(X)$ is a Banach space.

Proof. This follows easily from the fact that the set of finite Borel measures is sequentially complete, and $\|T\|$ depends continuously on $T \in \mathbf{M}_k(X)$, the remainder being entailed by the Banach-Steinhaus theorem, applied to $\mathcal{L}\big(\mathcal{B}_b(X)\hat{\otimes}_\pi \mathrm{Lip}(X)^{\hat{\otimes}_\pi k}, \mathbb{R}\big)$. □

Proposition 5.2.3. Let $g \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{R})$. Then $[\![g]\!] \in \mathbf{M}_k(\mathbb{R}^k)$, where $[\![g]\!]$ is defined as follows,

$$\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : [\![g]\!] \rangle = \int fg \det \pi' \, d\mathcal{L}^k$$

where $\pi = (\pi_1, \ldots, \pi_k)$. Moreover, $\|[\![g]\!]\| = |g| \cdot \mathcal{L}^k$.

5.2.4. First, recall the following commonplace concepts and results.

(i). If $1 < p \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then the weak topology on $\mathbf{L}^p(\mathbb{R}^k)$ is defined to be the coarsest locally convex topology making the linear functionals

$$\mathbf{L}^p(\mathbb{R}^k) \to \mathbb{C} : f \mapsto \int f \cdot g \, d\mathcal{L}^k \quad \text{continuous for all} \quad g \in \mathbf{L}^q(\mathbb{R}^k) .$$

With this topology, $\mathbf{L}^p(\mathbb{R})$ identifies with the dual space of $\mathbf{L}^q(\mathbb{R}^k)$, and the norm $\mathbf{L}^p(\mathbb{R}^k)$ identifies with the dual norm. In particular, the unit ball of $\mathbf{L}^p(\mathbb{R}^k)$ is weakly compact (it is metrisable since its topology has a countable base, seeing that $\mathbf{L}^q(\mathbb{R}^k)$ is separable).

(ii). We say that $f \in \mathbf{L}^1_{loc}(\mathbb{R}^k)$ is *weakly differentiable* if for all $j = 1, \ldots, k$ there exist $g_j \in \mathbf{L}^1_{loc}(\mathbb{R}^k)$, so that

$$\int f \cdot \partial_j \varphi \, d\mathcal{L}^k = -\int g_j \cdot \varphi \, d\mathcal{L}^k \quad \text{for all } 1 \leqslant j \leqslant k \, , \ \varphi \in \mathcal{D}(\mathbb{R}^k)$$

where $\mathcal{D}(\mathbb{R}^k)$ denotes the set of smooth compactly supported functions on $\mathbb{R}^k$. The g_j are uniquely determined up to equality $\mathcal{L}^k$ a.e., and are called the *weak partial derivatives* of f, $\partial_j f = g_j$.

(iii). For $1 \leqslant p \leqslant \infty$, let $\mathrm{W}^{1,p}(\mathbb{R}^k)$ be the *Sobolev* space of functions $f \in \mathbf{L}^p(\mathbb{R}^k)$ which are weakly differentiable with $\partial_j f \in \mathbf{L}^p(\mathbb{R}^k)$. Then $\mathrm{W}^{1,p}(\mathbb{R}^k)$ is a Banach space for the norm

$$\|f\|^p_{W^{1,p}} = \|f\|^p_p + \sum_{j=1}^k \|\partial_j f\|^p_p \ \text{ if } p < \infty \quad \text{and} \quad \|f\|_{W^{1,\infty}} = \max\bigl(\|f\|_\infty, \max_{j=1}^k \|\partial_j f\|_\infty\bigr) \, .$$

Here, we identify a.e. equal functions. In fact, $\mathrm{W}^{1,\infty}(\mathbb{R}^k)$ consists of true functions, as the following lemma and corollary show.

Lemma 5.2.5. Let $f, f_j \in \mathrm{Lip}(\mathbb{R}^k)$, $\mathrm{Lip}(f_j) \leqslant C$, and $f_j \to f$ point-wise. Then f_j, f have weak derivatives $\partial_i f_j, \partial_i f \in \mathbf{L}^\infty(\mathbb{R}^k)$, and

$$\int \partial_i f \cdot g \, d\mathcal{L}^k = \lim\nolimits_j \int \partial_i f_j \cdot g \, d\mathcal{L}^k \quad \text{for all } i = 1, \ldots, k, \, g \in \mathbf{L}^1(\mathbb{R}^k) \, .$$

Proof. Set

$$g^h_{ij}(x) = \frac{f_j(x + he_i) - f_j(x)}{h} \quad \text{for all } x \in \mathbb{R}^k , \, h > 0 \, .$$

Then $\|g^h_{ij}\|_\infty \leqslant \mathrm{Lip}(f_j) \leqslant C$, so $g^h_{ij} \in \mathbf{L}^\infty(\mathbb{R}^k)$, so (g^h_{ij}) is contained in a weak* compact subset of $\mathbf{L}^\infty(\mathbb{R}^k)$. Let $h_n \to 0$ be arbitrary, and fix $1 \leqslant i \leqslant n$, $j \in \mathbb{N}$. Let $h_{\alpha(n)}$ be any subsequence such that there exists $g_{ij} \in \mathbf{L}^1_{loc}(\mathbb{R}^k)$ such that $g_{ij} = \lim_n g_{ij}^{h_{\alpha(n)}}$ weakly in $\mathbf{L}^\infty(\mathbb{R}^k)$. Further passing to subsequences, we may assume that the convergence is also point-wise $\mathcal{L}^k$ a.e. Hence, we have $g_{ij} \in \mathbf{L}^\infty(\mathbb{R}^k)$, $\|g_{ij}\|_\infty \leqslant C$.

For $\varphi \in \mathcal{D}(\mathbb{R}^k)$, Lebesgue's theorem 1.5.7 shows

$$\begin{aligned}\int f\partial_i\varphi &= \lim_n \int f_j(x)\frac{\varphi(x+h_{\alpha(n)}e_i)-\varphi(x)}{h_{\alpha(n)}}\,dx\\ &= -\lim_n \int g_{ij}^{h_{\alpha(n)}}(x)\varphi(x+h_n^j e_i)\,dx = -\int g_{ij}\varphi\,.\end{aligned}$$

Hence, the class $g_{ij} \in \mathbf{L}^p(\mathbb{R}^k)$ is independent of the convergent subsequence chosen, and by compactness, we have convergence. Thus f_j is weakly differentiable, and $\partial_i f_j = g_{ij}$.

To prove that $\partial_i f_j \to \partial_i f$ weak* in $\mathbf{L}^\infty(\mathbb{R}^k)$, note that the ball of radius C in $\mathbf{L}^\infty(\mathbb{R}^k)$ is compact and metrisable in the $\sigma(\mathbf{L}^\infty, \mathbf{L}^1)$ topology. Moreover, the limit

$$\lim_j \int \partial_i f_j \cdot g\,d\mathcal{L}^k = \lim_j \lim_n \int g_{ij}^{h_n} \cdot g\,d\mathcal{L}^k$$

exists by Lebesgue's theorem 1.5.7; the limit

$$\lim_n \int \frac{f(x+h_n)-f(x)}{h_n} \cdot g\,d\mathcal{L}^k = \lim_n \lim_j \int g_{ij}^{h_n} \cdot g\,d\mathcal{L}^k$$

exists by the same argument as above.

By Grothendieck's double limit theorem [Bou87, IV.33, prop. 2], we may exchange limit order to obtain

$$\lim_j \int \partial_i f_j \cdot g\,d\mathcal{L}^k = \lim_n \int \frac{f(x+h_n)-f(x)}{h_n} \cdot g\,d\mathcal{L}^k = \int \partial_i f g\,d\mathcal{L}^k\,,$$

as required. □

Corollary 5.2.6. Endow $\mathrm{Lip}_b(\mathbb{R}^k)$ with the norm

$$\|f\|_{Lip} = \|f\| + \mathrm{Lip}(f) \quad \text{for all } f \in \mathrm{Lip}_b(\mathbb{R}^k)\,.$$

Then $\mathrm{Lip}_b(\mathbb{R}^k) = \mathrm{W}^{1,\infty}(\mathbb{R}^k)$, with equivalence of norms.

Proof. By lemma 5.2.5, any $f \in \mathrm{Lip}_b(\mathbb{R}^k)$ has essentially bounded weak derivatives.

Conversely, let $f \in \mathrm{W}^{1,\infty}(\mathbb{R}^k)$. Let $\eta \in \mathcal{D}(\mathbb{R}^k)$, $\eta \geqslant 0$, be such that $\int \eta\,d\mathcal{L}^k = 1$, and

$$f^\varepsilon(x) = \frac{1}{\varepsilon^n} \cdot \int \eta\Big(\frac{x-y}{\varepsilon}\Big) f(y)\,d\mathcal{L}^k(y) \quad \text{for all } \varepsilon > 0\,, x \in \mathbb{R}^k\,.$$

Then $f = \lim_{\varepsilon\to 0+} f^\varepsilon$ uniformly on compacts, and $\|f^\varepsilon\|_\infty \leqslant \|f\|_\infty$. Moreover, f^ε is differentiable, so

$$|f^\varepsilon(x) - f^\varepsilon(y)| \leqslant \|f^{\varepsilon\prime}\|_\infty \cdot \|x-y\| \leqslant \|f\|_{W^{1,\infty}} \cdot \|x-y\| \quad \text{for all } x,y \in \mathbb{R}^k .$$

Taking the limit $\varepsilon \to 0$, we find $f \in \mathrm{Lip}_b(\mathbb{R}^k)$. □

Proof of proposition 5.2.3. Clearly, $[\![g]\!]$ is multi-linear, and vanishes on $f\, d\pi_1 \wedge \cdots \wedge d\pi_k$ as soon as π_i is constant on a neighbourhood of $\{f \neq 0\}$, since in that case, $\det \varphi' = 0$ on $\{f \neq 0\}$. Moreover,

$$|\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : [\![g]\!]\rangle| \leqslant \int |fg \det \pi'|\, d\mathcal{L}^k \leqslant \prod_{j=1}^k \mathrm{Lip}(\pi_j) \cdot \int |f|\, d(|g| \cdot \mathcal{L}^k) ,$$

because by Hadamard's inquality,

$$|\det A| \leqslant \prod_{j=1}^k \|a_j\| \quad \text{for all } A = (a_1, \dots, a_k) \in \mathbb{R}^{k\times k} .$$

Let $\pi_j = \lim_i \pi_j^i$ point-wise, where $\mathrm{Lip}(\pi_j^i) \leqslant C$ for some $C > 0$. Then we have $\det \pi' = \lim_i \det \pi^{i\prime}$ weakly in $\mathbf{L}^\infty(\mathbb{R}^k)$ by lemma 5.2.5. Therefore,

$$\langle f\, d\pi_1^i \wedge \cdots \wedge d\pi_k^i : [\![g]\!]\rangle = \lim_i \langle f\, d\pi_1^i \wedge \cdots \wedge d\pi_k^i : [\![g]\!]\rangle \quad \text{for all } f \in \mathrm{Lip}_b(\mathbb{R}^k) ,$$

so $[\![g]\!] \in \mathbf{M}_k(\mathbb{R}^k)$.

We have seen that $\|[\![g]\!]\| \leqslant |g| \cdot \mathcal{L}^k$. Conversely, if $f \in \mathrm{Lip}_b(\mathbb{R}^k)$, we may assume $f \neq 0$. Then let $\chi_i \in \mathrm{Lip}_b(\mathbb{R}^k)$, $\chi_i = 0$ on $\{f = 0\}$, $\chi_i \to 1$ point-wise, and $|\chi_i| \leqslant 1$. Define $\pi_j^i = \mathrm{pr}_j$ for $j > 1$, $i \in \mathbb{N}$, and let

$$\pi_1^i(x) = \int_0^{x_1} \frac{\chi_i \cdot fg}{|fg|}(y, x_2, \dots, x_k)\, d\mathcal{L}^1(y) \quad \text{for all } x \in \mathbb{R}^k ,\ i \in \mathbb{N} .$$

Then it follows that

$$\langle f\, d\pi^i : [\![g]\!]\rangle = \int fg \det \pi^{i\prime}\, \mathcal{L}^k = \int \chi_i \cdot |f|\, d(|g| \cdot \mathcal{L}^k) \to \int |f|\, d(|g| \cdot \mathcal{L}^k) ,$$

proving the assertion. □

Definition 5.2.7. A current $T \in \mathbf{M}_k(X)$ is called *normal* if $\partial T \in \mathbf{M}_k(X)$. It is clear that ∂T satisfies condition (ii) in the definition of a current for any $T \in \mathbf{M}_k(X)$, and $(k+1)$-linearity is also obvious. The locality property for ∂T also holds automatically, as follows from the stronger locality established for T in part (iii) of the theorem below. Thus, for T to be normal, it is necessary and sufficient that ∂T have finite mass. The vector space of normal currents will be denoted $\mathbf{N}_k(X)$. Endowed with the norm defined by

$$\mathbf{N}_k(T) = \mathbf{M}_k(T) + \mathbf{M}_k(\partial T) \quad \text{for all } \ T \in \mathbf{N}_k(X) \ ,$$

the space of normal currents is a Banach space.

Theorem 5.2.8. Let $T \in \mathbf{M}_k(X)$. Its extension to $\mathcal{B}_b(X) \times \mathrm{Lip}(X)^k$ satisfies:

(i). T is $(k+1)$-linear, alternating in π_i ,

$$\langle d(f\pi_1) \wedge \cdots \wedge d\pi_k : T \rangle = \langle f \, d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle + \langle \pi_1 \, df \wedge \cdots \wedge d\pi_k : T \rangle \ ,$$

whenever $f, \pi_1 \in \mathrm{Lip}_b(X)$, and

$$\langle f \, d\psi_1(\pi) \wedge \cdots \wedge d\psi_k(\pi) : T \rangle = \langle f \det \psi'(\pi) \, d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle$$

whenever $\psi = (\psi_1, \dots, \psi_k)$, $\psi_j \in \mathcal{C}^1(\mathbb{R}^k)$, and ψ' is bounded;

(ii). for $f^i \to f$ in $\mathbf{L}^1(\|T\|)$, $\pi_j^i \to \pi_j$ point-wise, $\mathrm{Lip}(\pi_j^i) \leqslant C$ for some $C > 0$,

$$\langle f \, d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle = \lim_i \langle f^i \, d\pi_1^i \wedge \cdots \wedge d\pi_k^i : T \rangle \ ;$$

(iii). $\langle f \, d\pi_1 \wedge \cdots \wedge d\pi_k : T \rangle = 0$ whenever $\{f \neq 0\} = \bigcup_{j=1}^\infty B_j$ where the B_j are Borel, and for all j , there exists some i such that $\pi_i | B_j$ is constant.

5.2.9. We recall some concepts from the theory of the method of finite elements.

A *k-simplex* $\Delta \subset \mathbb{R}^k$ is the convex hull of $k+1$ affinely independent vectors from $\mathbb{R}^k$, called its *vertices*. The totality of vertices of Δ is denoted V_Δ (it is uniquely determined by Δ as the set of extreme points). A *face* is the convex hull of a subset of its vertices. Given a bounded, closed polyhedral domain $D \subset \mathbb{R}^k$, a *triangulation* is a finite family $\mathcal{T}$ of k-simplices, such that any two simplices intersect only in their boundaries, and any face of a simplex is either the face of another simplex, or countained in the boundary of D (note that in the literature, rather more general triangulations are considered). A family $\mathcal{S}$ of k-simplices is called *regular* if $\mathcal{S}$ is fine, and

$$\theta_{\mathcal{S}} = \sup\Bigl\{ \frac{\mathrm{diam}(\Delta)}{r_\Delta} \Bigm| \Delta \in \mathcal{S} \Bigr\} < \infty \quad \text{where} \quad r_\Delta = \sup\{2r | \exists x \ : \ B(x,r) \subset \Delta\} \ .$$

A family of triangulations $(\mathcal{T}_j)$ is regular if $\bigcup_j \mathcal{T}_j$ is.

Given a triangulation $\mathcal{T}$ of D , let $A_{\mathcal{T}}(D)$ be the set of functions $f : D \to \mathbb{R}$ which are affine when restricted to any $\Delta \in \mathcal{T}$. For any $\Delta \in \mathcal{T}$, and any vertex v of Δ , let $\alpha_{v,\Delta} : \mathbb{R}^k \to \mathbb{R}$ be the uniquely determined affine function such that $\alpha_{v,\Delta}(w) = \delta_{vw}$ for all vertices w of Δ . Because of the requirement on faces of simplices in triangulations, there exists a uniquely determined linear map

$$p_{\mathcal{T}} : \mathbb{R}^D \to A_{\mathcal{T}}(D) \quad \text{such that} \quad p_{\mathcal{T}}(f)|\Delta = p_\Delta(f) = \textstyle\sum_{v \in V_\Delta} f(v)\alpha_{v,\Delta} \text{ for all } \Delta \in \mathcal{T} .$$

Lemma 5.2.10. Let $(\mathcal{T}_\varepsilon)_{\varepsilon>0}$ be a regular family of triangulations of the bounded polyhedral domain D, where $\sup_{\Delta\in\mathcal{T}} \operatorname{diam}\Delta \leqslant \varepsilon$. Then, for some constant $\infty > C > 0$,

$$\|f - p_{\mathcal{T}_\varepsilon} f\|_{W^{1,\infty}} \leqslant C \cdot \varepsilon \cdot \|f\|_{W^{1,\infty}} \quad \text{for all } f \in \mathrm{W}^{1,\infty}(D^\circ) , \varepsilon > 0 .$$

Thus, any Lipschitz function on D can be approximated in $\mathrm{W}^{1,\infty}(D^\circ)$ by a sequence of piecewise affine functions.

Proof. Fix $\varepsilon > 0$ and $\Delta \in \mathcal{T}_\varepsilon$. Let $\hat{\Delta} = (\operatorname{diam}\Delta)^{-1} \cdot \Delta$. There exists an affine transformation $A : \hat{\Delta} \to S$, S denoting the standard k-simplex $S = \{x \in [0,1]^k | \sum_{j=1}^k x_j = 1\}$. The Jacobian of A has a finite upper bound independent of Δ and $\varepsilon > 0$, due to the regularity of $(\mathcal{T}_\varepsilon)$. Moreover, writing $\hat{f}(x) = f\big(\operatorname{diam}\Delta \cdot x\big)$,

$$\widehat{f - p_\Delta f} = (1 - p_{\hat{\Delta}})\hat{f} = (1 - p_S)(\hat{f} \circ A^{-1}) \circ A .$$

Thus, for some constant $0 < C < \infty$,

$$\|f - p_\Delta f\|_{W^{1,\infty}(\Delta^\circ)} \leqslant C\varepsilon \cdot \|1 - p_S\| \cdot \|f\|_{W^{1,\infty}(\Delta^\circ)} .$$

The assertion follows by taking sums over $\Delta \in \mathcal{T}_\varepsilon$. ∎

Proof of theorem 5.2.8. The continuity (ii) is fairly easy: by finiteness of mass and the boundedness assumption on the Lipschitz constants,

$$\langle f\, d\pi_1^\ell \wedge \cdots \wedge d\pi_k^\ell : T\rangle = \lim\nolimits_i \langle f^i\, d\pi_1^\ell \wedge \cdots \wedge d\pi_k^\ell : T\rangle \quad \text{uniformly in} \quad \ell.$$

Moreover, $\langle f^i, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle = \lim_\ell \langle f^i\, d\pi_1^\ell \wedge \cdots \wedge d\pi_k^\ell : T\rangle$ for all i, by the continuity axiom. By Moore's lemma [DS58, I.7.5, lemma 6], the diagonal limit exists and

$$\langle f\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle = \lim\nolimits_i \langle f^i, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle = \lim\nolimits_i \langle f^i, d\pi_1^i \wedge \cdots \wedge d\pi_k^i : T\rangle .$$

In particular, T is σ-additive in f. Thus, for (iii), replacing f by $1_{B_i} f$, we may assume π_i is constant on $\{f \neq 0\}$ for some i. Appropriately scaling π_j, we may assume $\operatorname{Lip}(\pi_j) \leqslant 1$ for all j. Seeking a contradiction, assume that there is a closed $C \subset \{f \neq 0\}$ and an $\varepsilon > 0$ such that $|\langle d\pi_1 \wedge \cdots \wedge d\pi_k : T \llcorner C\rangle| > \varepsilon$. Since $\|T\|$ is Borel regular by proposition 1.1.7, theorem 1.1.9 (i) gives $\delta > 0$ such that $\|T\|(C_\delta \setminus C) \leqslant \varepsilon$ where $C_\delta = \big\{x \in X \mid d(x, X) \leqslant \delta\big\}$.

Let $c = \pi_i$ on $\{f \neq 0\}$. Define, for $t \geqslant 0$, functions $g_t : X \to \mathbb{R}$ and $c_t : \mathbb{R} \to \mathbb{R}$ by

$$g_t(x) = \left(1 - \frac{3}{t}\operatorname{dist}(x, C)\right)^+ \text{ and } c_t(s) = (s-t)^+ + (t-s)_- \quad \text{where} \quad f_- = \min(f, 0)\,.$$

Let

$$T_{st} = |\langle g_s\, d\pi_1 \wedge \cdots \wedge d\pi_{i-1} \wedge d\pi_i^t \wedge d\pi_{i+1} \wedge \cdots \wedge d\pi_k : T\rangle|$$

where $\pi_i^t = c_t \circ (\pi_i - c) + c$. Since $g_0 = 1$ and $c_0 = \mathrm{id}$, we know that $T_{00} > \varepsilon$. By (ii), there exist $0 < t < s < \delta$ such that $T_{st} > \varepsilon$. (Note that $\operatorname{Lip}(c_t) = 1$.) Now, $\pi_i^t = c$ on C_t, and $\operatorname{supp} g_t \subset C_{t/2}$, so $T_{tt} = 0$. But this implies

$$T_{st} = T_{st} - T_{tt} \leqslant \int |g_s - g_t|\,\|T\| \leqslant \|T\|(C_\delta \setminus C) \leqslant \varepsilon\,,$$

since $|g_s - g_t| = g_s - g_t \leqslant g_s \leqslant 1$. This is a contradiction.

Therefore, approximating f by linear combinations of 1_C, $C \subset \{f \neq 0\}$ closed, we obtain the statement (iii).

We establish a special case of the chain rule in (i): For $\psi \in \mathcal{C}^1(\mathbb{R}) \cap \operatorname{Lip}(\mathbb{R})$, we have

$$\langle f\, d\pi_1 \wedge \cdots d\pi_{i-1} \wedge d(\psi(\pi_i)) \wedge \cdots \wedge d\pi_k : T\rangle = \langle f \psi'(\pi_i)\, d\pi_1 \wedge \cdots \wedge d\pi_k : T\rangle$$

for all $\pi_j \in \operatorname{Lip}(X)$, $f \in \mathcal{B}_b(X)$. Indeed, this follows from (iii) if ψ is constant; if ψ is linear, the statement is trivial; hence, we have the equality if ψ is affine. For ψ piecewise affine, the equality follows from (iii). Suffices to apply lemma 5.2.10 to obtain the general case.

Next, we prove that T is alternating in π_i. To simplify matters, assume that $k = 2$, and $\pi_1 = \pi_2 = \pi$. We need to prove $\langle f\, d\pi \wedge d\pi : T\rangle = 0$. Let

$$\sigma_k = \frac{1}{k} \cdot \varphi(k\pi) \quad \text{and} \quad \tau_k = \frac{1}{k}\varphi\Big(k\pi + \frac{1}{2}\Big)$$

where $\varphi \in \mathcal{C}^\infty(\mathbb{R})$, $\varphi = \mathrm{id}$ on $\mathbb{Z}$, $\varphi'(1+t) = \varphi'(t) \geqslant 0$ for all $t \in \mathbb{R}$, and $\varphi' = 0$ on $\left[0, \frac{1}{2}\right]$. Then $|\varphi(x) - x| \leqslant 1$, so (σ_k) and (τ_k) converge uniformly to π, with uniformly bounded Lipschitz constants (φ' is bounded). By the above,

$$\langle f\, d\sigma_k \wedge d\tau_k : T\rangle = \frac{1}{k^2} \cdot \big\langle f \varphi'(k\pi)\varphi\big(k\pi + \tfrac{1}{2}\big) : T\big\rangle = 0\,,$$

since $\varphi' = 0$ on $\left[0, \frac{1}{2}\right]$. By (ii), we find $\langle f\, d\pi \wedge d\pi : T\rangle = 0$, so T is alternating.

Now to the general case of the chain rule in (i). For ψ a linear function, the statement follows from the fact that T is multilinear and alternating in the π_i. Then from (iii), we find that it is also true if all components of ψ are simultaneously affine on each simplex of a triangulation of $\mathbb{R}^k$. The general case follows from lemma 5.2.10.

Now to the proof of the product rule in (i). Replacing T by $T \llcorner (d\pi_2 \wedge \cdots \wedge d\pi_k)$, we may assume $k = 1$. Let $S = (f, \pi_1)_\bullet(T)$; the identity reduces to

$$\langle d(g_1 g_2) : S\rangle = \langle g_1\, dg_2 : S\rangle + \langle g_2\, dg_1 : S\rangle \quad \text{for all } g_1, g_2 \in \mathcal{C}^1(\mathbb{R}^2) \cap \mathrm{Lip}_b(\mathbb{R}^2)$$

such that $g_j = \mathrm{pr}_j$ on a square $Q \supset (f, \pi_1)(X) \supset \mathrm{supp}\, S$. Let $(\mathcal{T}_\varepsilon)$ be regular family of triangulations for Q, and set $g_\varepsilon = p_{\mathcal{T}_\varepsilon}(g_1 g_2)$. Then $g_\varepsilon|\Delta = a_\varepsilon^\Delta\, \mathrm{pr}_1 + b_\varepsilon^\Delta\, \mathrm{pr}_2 + c_\varepsilon^\Delta$ on any $\Delta \in \mathcal{T}_\varepsilon$. From lemma 5.2.10, we find $g_\varepsilon \to g$ in $\mathrm{W}^{1,\infty}(Q)$; since $\partial_j(g_1 g_2) = g_i$ on Q if $\{i, j\} = \{1, 2\}$, we find that

$$\lim\nolimits_{\varepsilon\to 0} \sup\nolimits_{\Delta\in\mathcal{T}_\varepsilon} \sup\nolimits_{(x,y)\in\Delta} \big(|g_1(x,y) - a_\varepsilon^\Delta x| + |g_2(x,y) - b_\varepsilon^\Delta y|\big) = 0\,,$$

so by (ii) and (iii),

$$\langle d(g_1 g_2) : S\rangle = \lim_{\varepsilon\to 0} \sum_{\Delta\in\mathcal{T}_\varepsilon} \langle a_\varepsilon^\Delta\, dx : S \llcorner \Delta\rangle + \langle b_\varepsilon^\Delta\, dy : S \llcorner \Delta\rangle = \langle g_1\, dg_2 : S\rangle + \langle g_2\, dg_1 : S\rangle\,,$$

which finally proves the theorem. □

Corollary 5.2.11. Let $T \in \mathbf{M}_k(X)$ and $f \in \mathrm{Lip}_b(X)$. Then $\partial(T \llcorner f) = \partial T \llcorner f - T \llcorner df$.

Definition 5.2.12. Let $T_j, T \in \mathbf{M}_k(X)$. We say that $T_j \to T$ *weakly* if

$$\lim\nolimits_j \langle f\, d\pi : T_j\rangle = \langle f\, d\pi : T\rangle \quad \text{for all } f \in \mathrm{Lip}_b(X)\,,\ \pi = (\pi_1, \dots, \pi_k)\,,\ \pi_i \in \mathrm{Lip}(X)\,.$$

For any open subset $U \subset X$, the map $T \mapsto \|T\|(U)$ is l.s.c. for the topology of weak convergence on $\mathbf{M}_k(X)$. This follows from the equation $(**)$ in proposition 5.1.2, since this gives a representation of the mass as the supremum of weakly continuous functions.

Definition 5.2.13. For any $\mathcal{L}^k$ summable function $f \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{R})$, $U \subset \mathbb{R}^k$ open, define the *total variation* by

$$|\nabla f|(U) = \sup\Big\{ \int f \cdot \mathrm{div}\varphi\, d\mathcal{L}^k \;\Big|\; \varphi \in \mathcal{C}_c^{(1)}(U, \mathbb{R}^k)\,,\ |\varphi| \leqslant 1 \Big\} \quad \text{where} \quad \mathrm{div}\varphi = \sum_{j=1}^k \partial_j \varphi_j\,.$$

The summable function f is called of *bounded variation* if $|\nabla f|(\mathbb{R}^k) < \infty$. The vector space of BV functions is denoted $\mathrm{BV}(\mathbb{R}^k)$. It is a real Banach space for the norm

$$\|f\|_{BV} = \|f\|_1 + |\nabla f|(\mathbb{R}^k)\,.$$

Theorem 5.2.14. For any $T \in \mathbf{N}_k(\mathbb{R}^k)$, there exists a unique function $g \in \mathrm{BV}(\mathbb{R}^k)$ such that $T = [\![g]\!]$. Moreover, $\|\partial T\| = |\nabla g|$ on open subsets.

Lemma 5.2.15. Let μ be a continuous real linear functional on $\mathcal{B}_b(\mathbb{R}^k)$ which is weakly differentiable, i.e. there exists $\nabla\mu \in \mathbf{L}^1_{loc}(\mathbb{R}^k, \mathbb{R}^k)$, such that

$$\langle \mathrm{div}\varphi : \mu\rangle = -\int (\varphi : \nabla\mu)\, d\mathcal{L}^k \quad \text{for all } \varphi \in \mathcal{C}_c^{(1)}(\mathbb{R}^k, \mathbb{R}^k)\,.$$

Then there exists $g \in \mathrm{BV}(\mathbb{R}^k)$ such that $\langle f : \mu \rangle = \int fg \, d\mathcal{L}^k$ for all $f \in \mathcal{B}_b(\mathbb{R}^k)$.

Proof. Let $\chi \in \mathcal{C}_c^{(\infty)}(\mathbb{R}^k)$, $0 \leqslant \chi \leqslant 1$, $\int \chi \, d\mathcal{L}^k = 1$. Set $\chi_\varepsilon(x) = \varepsilon^{-k} \cdot \chi(\varepsilon^{-1}x)$ for $x \in \mathbb{R}^k$, $\varepsilon > 0$. Define $g_\varepsilon(x) = \langle \chi_\varepsilon(x - \sqcup) : \mu \rangle$. Then

$$\int \varphi g_\varepsilon \, d\mathcal{L}^k = \langle f_\varepsilon * \varphi : \mu \rangle \quad \text{for all } \varphi \in \mathcal{C}_c^{(1)}(\mathbb{R}^k),$$

so (g_ε) is bounded in $\mathrm{BV}(\mathbb{R}^k)$, and $g_\varepsilon \cdot \mathcal{L}^k \to \mu$ in the weak topology on $\mathcal{B}_b(\mathbb{R}^k)'$. Moreover, the inclusion of $\mathrm{BV}(\mathbb{R}^k)$ in $\mathbf{L}^1_{loc}(\mathbb{R}^k)$ is compact, so some subsequence of (g_ε) converges in $\mathbf{L}^1_{loc}(\mathbb{R}^k)$ to $g \in \mathrm{BV}(\mathbb{R}^k)$. □

Proof of theorem 5.2.14. Define a continuous functional by

$$\langle f : \mu \rangle = \langle f \, dx_1 \wedge \cdots \wedge dx_k : T \rangle \quad \text{for all } f \in \mathcal{B}_b(\mathbb{R}^k).$$

By the lemma, suffices to prove the weak differentiability of μ.

Let (e_j) be an orthonormal basis of $\mathbb{R}^k$, and let $\pi_j(x) = (x : e_j) \cdot e_j$. For $\varphi \in \mathcal{C}_c^{(1)}(\mathbb{R}^k)$, we have

$$|\langle \partial_i \varphi : \mu \rangle| = |\langle d\varphi \wedge d\pi_1 \wedge \cdots \wedge d\pi_{i-1} \wedge d\pi_{i+1} \wedge \cdots \wedge d\pi_k : T \rangle| \leqslant \int |\varphi| \, d\|\partial T\|$$

by the chain rule in theorem 5.2.8. This implies $\mu = g \cdot \mathcal{L}^k$ for some $g \in \mathrm{BV}(\mathbb{R}^k)$, and $|\nabla g| \leqslant \|\partial T\|$. Again applying the chain rule, we obtain $[\![g]\!] = T$. By a similar device as in the proof of proposition 5.2.3, we find $\|\partial T\| \leqslant |\nabla g|$. □

Corollary 5.2.16. Let X be separable, $T \in \mathbf{N}_k(X)$. For any $\mathcal{L}^k$ neglible Borel $B \subset \mathbb{R}^k$,

$$\|T \llcorner d\pi\|(\pi^{-1}(B)) = 0 \quad \text{for all } \pi \in \mathrm{Lip}(X, \mathbb{R}^k).$$

Moreover, $\|T\| \ll \mathcal{H}^k$. (Here, we use the usual notation for absolute continuity, although $\mathcal{H}^k$ is usually not finite on bounded subsets.)

Proof. Let $f \in \mathcal{B}_b(X)$. Then $\pi_\bullet(T \llcorner f) \in \mathbf{N}_k(\mathbb{R}^k)$, since $\partial(\pi_\bullet(S)) = \pi_\bullet(\partial S)$. By theorem 5.2.14, $\pi_\bullet(T \llcorner f) = [\![g]\!]$ for some $g \in \mathrm{BV}(\mathbb{R}^k)$. Then, x_j denoting the jth coordinate function on $\mathbb{R}^k$,

$$\begin{aligned} \langle f 1_{\pi^{-1}(N)} : T \llcorner d\pi \rangle &= \langle 1_{\pi^{-1}(N)} : T \llcorner (f \, d\pi) \rangle = \langle 1_N : \pi_\bullet(T \llcorner f \, d\pi) \rangle \\ &= \langle 1_N \, dx_1 \wedge \cdots \wedge dx_k : \pi_\bullet(T \llcorner f) \rangle = \int_N g \, d\mathcal{L}^k = 0. \end{aligned}$$

Since f was arbitrary, from proposition 5.1.2 we conclude that $\|T \llcorner d\pi\|(\pi^{-1}(N)) = 0$.

Finally, if $\mathcal{H}^k(M) = 0$ where $M \subset X$, then since $\mathcal{H}^k$ is Borel regular, $M \subset B$ where $B \in \mathcal{B}(X)$ is $\mathcal{H}^k$-negligible. Thus, $\mathcal{H}^k(\pi(B)) = 0$ by theorem 2.2.2. By Borel regularity, $\pi(B) \subset N$, for $N \in \mathcal{B}(\mathbb{R}^k)$ $\mathcal{L}^k$-negligible. Then $M \subset \pi^{-1}(N)$, so $\|T \llcorner d\pi\|(M) = 0$. Since π was arbitrary, proposition 5.1.2 shows that $\|T\|(M) = 0$. □

5.3 Rectifiable Currents

Definition 5.3.1. Let $T \in \mathbf{M}_k(X)$. Then T is called a *rectifiable current* if its mass $\|T\|$ is k-rectifiable. A rectifiable current is called *integer-rectifiable* if for all $\varphi \in \mathrm{Lip}(X, \mathbb{R}^k)$, and any open $U \subset X$, $\varphi_\bullet(T \llcorner U) = [\![\vartheta]\!]$ for some $\mathcal{L}^k$-summable and integer-valued $\vartheta : \mathbb{R}^k \to \mathbb{Z}$. Finally, a rectifiable current is called *integral* if it is integer-rectifiable and normal. Denote the spaces of rectifiable, integer-rectifiable, and integral currents, respectively, by

$$\mathcal{R}_k(X)\,,\ \mathcal{I}_k(X)\,,\quad \text{and}\quad \mathbf{I}_k(X)\,.$$

The two former spaces are closed in $\mathbf{M}_k(X)$, whence the latter is closed in $\mathbf{N}_k(X)$.

Remark 5.3.2. If $T \in \mathbf{N}_k(X)$ and $\|T\|$ is concentrated on a countably k-rectifiable set, then corollary 5.2.16 shows that $T \in \mathcal{R}_k(X)$.

Theorem 5.3.3. Let $T \in \mathbf{M}_0(X)$, and assume X that is complete and contains a dense subset whose cardinality is an Ulam number (e.g. X separable and complete).

(i). We have $T \in \mathcal{I}_0(X)$ if and only if $\langle 1_U : T\rangle \in \mathbb{Z}$ for all open subset $U \subset X$.

(ii). We have $T \in \mathcal{I}_0(X)$ if and only if $\varphi_\bullet T \in \mathcal{I}_0(\mathbb{R})$ for all $\varphi \in \mathrm{Lip}(X, \mathbb{R})$.

(iii). If $X = \mathbb{R}^n$, then $T \in \mathcal{R}_0(X)$ if and only if $\varphi_\bullet T \in \mathcal{R}_0(\mathbb{R})$ for all $\varphi \in \mathrm{Lip}(X, \mathbb{R})$.

Proof of (i). The condition is certainly necessary for integer-rectifiability. Assume that it is given. Then let

$$A = \big\{x \in X \bigm| \|T\|B(x,r) \geqslant 1 \text{ for all } r > 0\big\}\,.$$

Clearly, A is closed. Since $\|T\|$ is a finite measure, for any $r > 0$, A is covered by finitely many balls of radius r. Hence, A is compact. By essentially the same argument, A is also discrete. Thus, A is finite. Of course, $T \llcorner A \in \mathcal{R}_0(X)$, and by continuity of T, we find $T \llcorner A \in \mathcal{I}_0(X)$. Suffices to show that T is supported on A. If $x \in X \setminus A$, there is a ball $B \subset X \setminus A$ such that $\|T\|(B) < 1$. Then, for any open $U \subset B$, $|\langle 1_U : T\rangle|$ is a natural number less than 1, and thus equals 0. By proposition 5.1.2, $\|T\|(B) = 0$. If $K \subset X \setminus A$ is compact, then $\varepsilon = \mathrm{dist}(X \setminus A, K) > 0$. Since K is contained in the union of finitely many balls of radius $< \frac{\varepsilon}{2}$, we find $\|T\|(K) = 0$. Since $\|T \llcorner (X \setminus A)\|$ is concentrated on a σ-compact by corollary 1.2.6, we find that $T = T \llcorner A$.

Proof of (ii). Again, the condition is clearly necessary, and we assume that it obtains. Let $U \subset X$ be open, and denote $\varphi = \mathrm{dist}(\sqcup, X \setminus U)$. Then $\varphi \in \mathrm{Lip}(X, \mathbb{R})$ by the triangle inequality, and

$$\mathbb{Z} \ni \langle 1_{]0,\infty[} : \varphi_\bullet T\rangle = \langle 1_{\varphi>0} : T\rangle = \langle 1_U : T\rangle\,,$$

so by (i), we find $T \in \mathcal{I}(X)$.

Proof of (iii). The statement follows from the following lemma. □

Lemma 5.3.4. Let μ be a signed measure on $\mathbb{R}^k$, i.e. $\mu = \nu \llcorner f$ where ν is a Radon measure and $f : \mathbb{R}^k \to \mathbb{R}$ is a locally ν summable function. Furthermore, define

$$f_{x,\xi}(y) = \max_{i=1}^{k} \xi_i \cdot |x_i - y_i| \quad \text{for all } y \in \mathbb{R}^k\,,\ Q = \mathbb{Q}^k \times \mathbb{Q}^k_{>0}\,.$$

Then μ is 0-rectifiable (i.e. concentrated on a countable set and absolutely continuous to $\mathcal{H}^0$) if and only if this is the case for the image measures $f_{x,\xi}(\mu)$, for all $(x,\xi) \in Q$.

Proof. The condition is necessary, so assume it given. By subtracting its point masses, we may assume that μ has no point masses. Seeking a contradiction, assume that μ is not 0-rectifiable, and let $0 \leqslant \ell \leqslant k$ be minimal with the property

$$|\mu|(x_0 + \mathbb{R}^{N \setminus I}) > 0 \quad \text{for some } I \subset \{1, \ldots, k\}\,,\ \#I = k - \ell\,,\ x_0 \in \mathbb{R}^k\,.$$

Since μ has no point masses, $\ell \geqslant 1$. Possibly replacing μ by $-\mu$, there exists $\varepsilon > 0$ and $x_1 \in \mathbb{Q}^k$ such that

$$\mu(M) > 3\varepsilon \quad \text{where} \quad M = \big\{ y \in x_0 + \mathbb{R}^I \bigm| \|y - x_1\|_\infty < 1 \big\}\,.$$

Note that M only depends on the components x_{1i}, $i \notin I$, of x_1, so we may choose x_{1i}, $i \in I$, arbitrarily close to x_{0i} without modifying M. Let $k \in \mathbb{N}$ be sufficiently large such that

$$|\mu|(N_k) < \varepsilon \quad \text{where} \quad N_k = \big\{ y \in \mathbb{R}^k \bigm| \operatorname{dist}_\infty(y, M) \in \left]0, \tfrac{2}{k}\right[\big\}\,.$$

The existence of such a k follows since M has finite measure, and $M = \bigcap_{k=1}^\infty (M \cup N_k)$.

Now, modify x_{1i} such that $\max_{i \in I} |x_{0i} - x_{1i}| < \frac{1}{k}$. Define $\xi \in \mathbb{Q}^k_{>0}$ by $\xi_i = k$ for $i \in I$ and $\xi_i = 1$ otherwise. Then

$$M \subset f^{-1}_{x_1,\xi}([0,1[) \subset M \cup N_k\,.$$

Indeed, for $y \in M$,

$$\xi_i \cdot |x_{1i} - y_i| = k \cdot |x_{1i} - x_{0i}| < 1 \quad \text{if} \quad i \in I\,,$$

and $\xi_i \cdot |x_{1i} - y_i| \leqslant \|y - x_1\|_\infty < 1$ otherwise. For $f_{x_1,\xi}(y) < 1$ and $y \notin M$, we have

$$|y_i - x_{0i}| \leqslant |y_i - x_{1i}| + |x_{1i} - x_{0i}| < \frac{2}{k} \quad \text{for all } i \in I\,.$$

Moreover, $|y_i - x_{1i}| < 1$ for all $i \notin I$, so $z \in \mathbb{R}^k$, defined by $z_i = x_{0i}$ for $i \in I$, and $z_i = y_i$ for all $i \notin I$, is contained in M. Thus, $\|y - z\|_\infty < \frac{2}{k}$, and $y \in N_k$.

Let $A \subset \mathbb{R}$ be countable, such that $\tilde{\mu} = f_{x_1,\xi}(\mu)$ is concentrated on A. For any $r \in \mathbb{R}$, $f^{-1}_{x_1,\xi}(r)$ is contained the union of finitely many affine hyperplanes. Thus, by the minimality of ℓ, we conclude $|\mu|(M \cap f_{x_1,\xi}^{-1}(r)) = 0$ for all r. Therefore,

$$|\tilde{\mu}|([0,1[) = |\mu|\big(f_{x_1,\xi}^{-1}(A \cap [0,1[)\big) \leqslant |\mu|(N_k) < \varepsilon\,.$$

On the other hand,

$$\tilde{\mu}([0,1[) = \mu\big(f_{x_1,\xi}^{-1}([0,1[)\big) = \mu(M) + \mu(f_{x_1,\xi}^{-1}([0,1[\cap N_k) \geqslant \mu(M) - |\mu|(N_k) > 2\varepsilon\,,$$

a contradiction. □

Theorem 5.3.5. Let $T \in \mathbf{M}_k(X)$. Then $T \in \mathcal{R}_k(X)$ (resp. $T \in \mathcal{I}_k(X)$) if and only if there exist compacts $K_j \subset \mathbb{R}^k$, functions $\vartheta_j \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{R})$ (resp. $\vartheta_j \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{Z})$) such that $\operatorname{supp}\vartheta_j \subset K_j$, and Lipschitz maps $\varphi_j : K_j \to X$, such that

$$T = \sum_{k=0}^{\infty} \varphi_{j\bullet}[\![\vartheta_j]\!] \quad \text{and} \quad \mathbf{M}(T) = \sum_{k=0}^{\infty} \mathbf{M}\big(\varphi_{j\bullet}[\![\vartheta_j]\!]\big)\,.$$

Moreover, if X is the dual of a separable Banach space, T is the limit in $\mathbf{M}_k(X)$ of normal currents.

Proof. The condition is clearly sufficient for the rectifiable case. For the integer-rectifiable case, it suffices to check that $S = \varphi_\bullet[\![\vartheta]\!]$ is integer-rectifiable for $\vartheta \in \operatorname{Lip}(K, X)$, $K \subset X$ compact, and $\vartheta \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{Z})$, $\operatorname{supp}\vartheta \subset K$. Rectifiability is clear, and for any open $U \subset X$, $\psi \in \operatorname{Lip}(X, \mathbb{R}^k)$, setting $\phi = \psi \circ \varphi$,

$$\begin{aligned}
\langle f\,d\pi : \psi_\bullet(S \llcorner U)\rangle &= \int_{K\cap\varphi^{-1}(U)} (f\circ\phi)\det(\pi\circ\phi)'\vartheta\,d\mathcal{L}^k \\
&= \int_{K\cap\varphi^{-1}(U)} (f\cdot\det\pi')\circ\phi\cdot\det\phi'\vartheta\,d\mathcal{L}^k \\
&= \int_{\phi(K)\cap U} f\det\pi'\cdot(\vartheta\cdot\operatorname{sgn}\det\phi')\circ\phi^{-1}\cdot N(\phi|K\cap\varphi^{-1}(U),\sqcup)\,d\mathcal{L}^k\,,
\end{aligned}$$

by the change of variables formula (corollary 4.1.6) and uniqueness of measure in $\mathbb{R}^k$ (theorem 2.4.3). Hence, $\psi_\bullet(S \llcorner U) = [\![(\vartheta\cdot\operatorname{sgn}\det\phi')\circ\phi^{-1}\cdot N(\phi|K\cap\varphi^{-1}(U),\sqcup)]\!]$, and sufficiency also obtains in the integer-rectifiable case.

Conversely, let T be rectifiable, and let $S \subset X$ be a countably k-rectifiable subset on which $\|T\|$ is concentrated. By lemma 4.2.3, $S \setminus \bigcup_{j=0}^{\infty}\varphi_j(K_j)$ is $\mathcal{H}^k$-negligible for some compacts $K_j \subset \mathbb{R}^k$, some bi-Lipschitz maps $f_j : K_j \to f_j(K_j) = S_j \subset S$, such that S_j are pairwise disjoint. Let $T_j = \varphi_{j\bullet}^{-1}(T \llcorner S_j)$. Then $\|T_j\|$ is absolutely continuous to $\mathcal{H}^k$, by theorem 2.2.2. By theorem 1.9.5, the functional μ_j on $\mathcal{B}_b(\mathbb{R}^k)$, defined by

$$\langle f : \mu\rangle = \langle f\,dx_1\wedge\cdots\wedge dx_n : R_j\rangle \quad \text{for all } f \in \mathcal{B}_b(\mathbb{R}^k)\,,$$

is a signed measure, $\mu_j = \vartheta_j\cdot\mathcal{L}^k$, for some $\vartheta_j \in \mathbf{L}^1(\mathbb{R}^k)$ vanishing outside K_j. By the chain rule in theorem 5.2.8, we find $R_j = [\![\vartheta_j]\!]$. Then $T \llcorner S_j = \varphi_{j\bullet}[\![\vartheta_j]\!]$ follows by locality in theorem 5.2.8, and the assertion in the rectifiable case follows from disjointness of S_j. In the integer-recitifiable case, we may assume ϑ_j integer-valued by definition.

If X is the dual of a separable Banach space, we may assume that the φ_j are defined on all of $\mathbb{R}^k$. Replacing φ_j by $\varphi_j(\mathrm{Lip}(\varphi_j)^{-1}\llcorner)$, we may assume $\mathrm{Lip}(\varphi_j) \leqslant 1$. Fix $\varepsilon > 0$ and choose $\theta_j \in \mathrm{BV}(\mathbb{R}^k)$, $\|\vartheta_j - \theta_j\|_1 \leqslant \frac{\varepsilon}{2^{j+1}}$. Then $S = \sum_{j=0}^{\infty} \varphi_{j\bullet}[\![\theta_j]\!]$ is the limit of normal currents, and

$$\mathbf{M}(T-S) \leqslant \sum_{j=0}^{\infty} \mathrm{Lip}(\varphi_j) \cdot \mathbf{M}([\![\vartheta_j]\!] - [\![\theta_j]\!]) \leqslant \sum_{j=0}^{\infty} \int \big||\vartheta_j| - |\theta_j|\big| \, \mathcal{L}^k \leqslant \sum_{j=0}^{\infty} \frac{\varepsilon}{2^{j+1}} = \varepsilon\,,$$

proving the assertion. □

Proposition 5.3.6. Let $T \in \mathcal{R}_k(X)$, and define $S_T = \{\Theta_{k*}(\|T\|, \llcorner) > 0\}$. Then S_T is countably k-rectifiable, $\|T\|$ is concentrated on S_T, and for any Borel $S \subset X$ on which $\|T\|$ is concentrated, $\mathcal{H}^k(S_T \setminus S) = 0$.

Proof. Let S be countably k-rectifiable and $\theta : S \to [0, \infty[$ be $\mathcal{H}^k \llcorner S$ summable, such that $\|T\| = \vartheta \cdot \mathcal{H}^k \llcorner S$. By the non-trivial density result [Kir94, th. 5.4.], $\vartheta(x) = \Theta_{*k}(\|T\|, x)$ for $\mathcal{H}^k$-a.e. $x \in S$. Thus,

$$\mathcal{H}^k(S_T \setminus S) = \mathcal{H}^k(S_T \setminus (S \cap \{\theta > 0\})) = 0\,,$$

proving the assertion. □

Definition 5.3.7. The previous proposition allows us to define the *size* of $T \in \mathcal{R}_k(X)$ as $\mathbf{S}(T) = \mathcal{H}^k(S_T)$.

5.4 Normal Currents and Slicing

In what follows, we shall always assume X to be complete, and to contain a dense subset whose cardinality is an Ulam number (e.g., X separable and complete). This assumption guarantees that any finite Borel measure on X be concentrated on a σ-compact, by corollary 1.2.6.

On our way to the Plateau problem, we exhibit various compactness properties of different classes of currents, the first of which the following equicontinuity of normal currents.

Proposition 5.4.1. Let $T \in \mathbf{N}_k(X)$. Then, for $f \in \mathrm{Lip}_b(X)$ and $\sigma_i, \tau_i \in \mathrm{Lip}_1(X)$, we have

$$\big|\langle f\, d\sigma : T\rangle - \langle f\, d\tau : T\rangle\big| \leqslant \sum_{i=1}^{k} \left[\int_X |f \cdot (\sigma_i - \tau_i)| \, d\|\partial T\| + \mathrm{Lip}(f) \cdot \int_{\operatorname{supp} f} |\sigma_i - \tau_i| \, d\|T\| \right].$$

In particular, bounded subsets of $\mathbf{N}_k(X)$ are equicontinuous on $\mathrm{Lip}_b(X) \times \mathrm{Lip}_1(X)^k$.

Proof. It suffices to prove the assertion for bounded σ_i and τ_i, since the general case then follows by restricting T to sets on which σ_i, τ_i take their values in $[k, k+1[$ for $k \in \mathbb{Z}$.

So, assume σ_i, τ_i bounded, and set $\sigma' = (\sigma_2, \dots, \sigma_k)$. Then

$$\begin{aligned}\langle f\, d\sigma : T\rangle &= \langle f\, d\sigma' : T \llcorner\, d\sigma_1\rangle = \langle f\, d\sigma' : \partial T \llcorner\, \sigma_1 - \partial(T \llcorner\, \sigma_1)\rangle \\ &= \langle f\sigma_1\, d\sigma' : \partial T\rangle - \langle \sigma_1\, df \wedge d\sigma' : T\rangle\,,\end{aligned}$$

by corollary 5.2.11. We obtain

$$\big|\langle f\, d\sigma : T\rangle - \langle f\, d\tau_1 \wedge d\sigma' : T\rangle\big| \leqslant \int_X \big|f\cdot(\sigma_1 - \tau_1)\big|\, d\|\partial T\| + \int_{\operatorname{supp} f} |\sigma_1 - \tau_1|\, d\|T\|\,,$$

because $\langle (\tau_1 - \sigma_1)\, df \wedge d\sigma' : T\rangle = 0$ for $\sigma_1 = 0$ on $X \setminus \operatorname{supp} f$, due to locality. Applying the alternating property and the triangle inequality, we obtain our claim. □

Compactness of Normal Currents 5.4.2. Let $(T_j) \subset \mathbf{N}_k(X)$ be a bounded sequence. Assume that for every $j \in \mathbb{N}$, there exists a compact $K_j \subset X$, such that

$$\big(\|T_i\| + \|\partial T_i\|\big)(X \setminus K_j) \leqslant \frac{1}{j} \quad \text{for all } i \in \mathbb{N}\,.$$

Then a subsequence of (T_j) converges to $T \in \mathbf{N}_k(X)$ such that $\|T\| + \|\partial T\|$ is concentrated on $\bigcup_{j=0}^{\infty} K_j$.

Proof. Passing to a subsequence, we may assume there are finite Borel measures μ and ν on X such that $\mu = \lim_j \|T_j\|$ and $\nu = \lim_j \|\partial T_j\|$ weak* in $\mathcal{C}_b(X)'$. Since $(\mu+\nu)(X \setminus K_j) \leqslant \frac{1}{j}$, we find that $\mu + \nu$ is concentrated on $\bigcup_{j=0}^{\infty} K_j$.

We claim that (T_j) has a pointwise convergent subsequence. By a diagonal argument, it suffice to find for each $p \in \mathbb{N} \setminus 0$, a subsequence $\alpha = \alpha_p$, such that

$$\limsup_{nm} \big|\langle f\, d\pi : T_{\alpha(n)} - T_{\alpha(m)}\rangle\big| \leqslant \frac{3}{p} \quad \text{for all } f\, d\pi \in \mathcal{D}^k(X)\,,\ f, \pi_i \in \operatorname{Lip}_1(X)\,,\ |f| \leqslant p\,.$$

To that end, let $\chi \in \operatorname{Lip}_1(X)$, $1_{K_{2p^2}} \leqslant \chi \leqslant 1$. By locality, $T_j - T_j \llcorner\, \chi = T_j \llcorner\, (1-\chi)$, so

$$\mathbf{N}(T_j - T_j \llcorner\, \chi) \leqslant \mathbf{N}(T_j \llcorner\, (X \setminus K_{2p^2})) \leqslant \frac{1}{p^2}\,.$$

To establish our claim, it suffices prove that for a convenient α, $\langle f\, d\pi : T_{\alpha(j)} \llcorner\, \chi\rangle$ converges for all $f, \pi_i \in \operatorname{Lip}_1(X)$.

Let $Z = \operatorname{Lip}_1\big(\bigcup_{j=0}^{\infty} K_j\big)$. Then Z, endowed with the topology of uniform convergence on compact subsets, is a separable metrisable space. Select a countable dense subset $D \subset Z$, and a subsequence α, such that $\langle f\, d\pi : T_{\alpha(j)} \llcorner\, \chi\rangle$ converges for all $f, \pi_i \in D$. Let $f, \pi_i \in \operatorname{Lip}_1(X)$. Then, for any $h, \sigma_i \in Z$, proposition 5.4.1 gives

$$\limsup_{mn} \big|\langle f\, d\pi : T_{\alpha(n)} \llcorner\, \chi - T_{\alpha(m)} \llcorner\, \chi\rangle\big| \leqslant 2 \limsup_n \big|\langle f\chi\, d\pi : T_n\rangle - \langle h\chi\, d\sigma : T_n\rangle\big|$$

$$\leqslant \limsup_n \Big[\int \chi |f-h|\, d\|T_j\| + \sum_{i=1}^k \int_{\operatorname{supp}\chi} (|f|+1)|\pi_i - \sigma_i|\, d\big(\|T_n\| + \|\partial T_n\|\big) \Big]$$

$$\leqslant \int \chi |f-h|\, d\mu + \sum_{i=1}^k \Big[\int_{\operatorname{supp}\chi} (|f|+1)|\pi_i - \sigma_i|\, d\nu + \int_{\operatorname{supp}\chi} |\pi_i - \sigma_i|\, d\mu \Big]$$

Approximating f, π_i uniformly on compact subsets of $\bigcup_{j=0}^\infty K_j$ by h, σ_i, our claim follows.

Now define $\langle \omega : T\rangle = \lim_n \langle \omega : T_{\alpha(n)}\rangle$ for all $\omega \in \mathcal{D}^k(X)$. Then T is a $(k+1)$-linear and local current, such that T and ∂T have finite mass $\|T\| \leqslant \mu$, $\|\partial T\| \leqslant \nu$. We need to check the continuity axiom (ii). By finiteness of mass, it suffices to check it for $f \in \operatorname{Lip}_b(X)$ of bounded support. Then proposition 5.4.1 gives

$$\big|\langle f\, d\pi : T\rangle - \langle f\, d\sigma : T\rangle\big| \leqslant \sum_{i=1}^k \Big[\int \big|f \cdot (\pi_i - \sigma_i)\big|\, d\mu + \operatorname{Lip}(f) \cdot \int_{\operatorname{supp} f} |\pi_i - \sigma_i|\, \nu \Big]$$

for all $\pi_i, \sigma_i \in \operatorname{Lip}_1(X)$. This immediately gives the continuity. □

5.4.3. The next step is to introduce the so-called *slicing technique*: Given a k-dimensional normal current T, its contraction $T \llcorner d\pi$ for $\pi : X \to \mathbb{R}^m$ can be represented as the integral of $(k-m)$-dimensional normal currents ('slices') concentrated on the fibres of π. This technique is important for proofs by induction on the dimension of currents.

Slicing Theorem 5.4.4. Let $T \in \mathbf{N}_k(X)$, let $L \subset X$ be a σ-compact on which T and ∂T concentrated, and let $\pi : X \to \mathbb{R}^m$ be Lipschitz, where $m \leqslant k$.

(i). For all $x \in \mathbb{R}^m$, there exist $\langle T, \pi, x\rangle \in \mathbf{N}_{k-m}(X)$, such that $\langle T, \pi, x\rangle$ and $\partial\langle T, \pi, x\rangle$ are concentrated on $L \cap \pi^{-1}(x)$, $x \mapsto \|\langle T, \pi, x\rangle\|$ is weakly $\mathcal{L}^m$ measurable,

$$\|T \llcorner d\pi\| = \int_{\mathbb{R}^m} \|\langle T, \pi, x\rangle\|\, \mathcal{L}^m(x) \quad \text{weakly in} \quad \mathcal{K}(\mathbb{R}^m)' , \tag{$*$}$$

$x \mapsto \langle T, \pi, x\rangle$ is weakly $\mathcal{L}^m$ measurable, and for all $\varphi \in \mathcal{K}(\mathbb{R}^m)$,

$$T \llcorner (\varphi \circ \pi\, d\pi) = \int_{\mathbb{R}^m} \langle T, \pi, x\rangle \varphi(x)\, d\mathcal{L}^m(x) \quad \text{weakly in} \quad \mathrm{MF}_k(X) . \tag{$**$}$$

(ii). Whenever $L' \subset X$ is a σ-compact, and $T_x \in \mathbf{M}_{k-m}(X)$ are concentrated on L', satisfy $(**)$ with T_x in place of $\langle T, \pi, x\rangle$, and $x \mapsto \mathbf{M}(T_x)$ is $\mathcal{L}^m$ summable, then for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$, $T_x = \langle T, \pi, x\rangle$.

(iii). For $m = 1$, then for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$,

$$\langle T, \pi, x\rangle = \lim_{y \to x+} \frac{T \llcorner (1_{\{x<\pi<y\}}\, d\pi)}{y - x} = (\partial T) \llcorner \{\pi > x\} - \partial\big(T \llcorner \{\pi > x\}\big) .$$

The proof requires a series of lemmata. The first of these is conventionally termed the 'localisation lemma'.

Lemma 5.4.5. Let $T \in \mathbf{N}_k(X)$ and $\varphi \in \operatorname{Lip}(X)$. Let $S_t = T \llcorner \{\varphi > t\}$ for $t \in \mathbb{R}$ and fix a σ-compact L on which T and ∂T are concentrated. Then, for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$,

$$(\partial T) \llcorner \{\varphi > t\} - \partial S_t = \frac{\mathrm{d}}{\mathrm{d}t}(S_t \llcorner d\varphi), \; \|\partial S_t\|(\varphi^{-1}(t)) \leqslant \frac{\mathrm{d}}{\mathrm{d}t}\|T \llcorner d\varphi\|\{\varphi \leqslant t\},$$

and S_t and ∂S_t are concentrated on L.

Proof. Let $\mu = \|T\| + \|\partial T\|$. Fix disjoint compacts (K_j) with μ concentrated on $\bigcup_j K_j$. Define

$$g_j(t) = \mu\big(K_j \cap \{\varphi \leqslant t\}\big) \quad \text{and} \quad g(t) = \sum_{j=0}^{\infty} g_j(t) = \mu\{\varphi \leqslant t\}.$$

Since g, g_j, and $t \mapsto f(t) = \|T \llcorner d\varphi\|\{\varphi \leqslant t\}$ are increasing, the set

$$T = \left\{ t \in \mathbb{R} \;\middle|\; g'(t), g_j'(t), f'(t) \text{ exist }, g'(t) = \sum_{j=0}^{\infty} g_j'(t) < \infty \right\}$$

has full $\mathcal{L}^1$ measure. Fix $t \in T$, and let $\varepsilon_\ell \to 0$. Let $f_\ell : \mathbb{R} \to [0,1]$ be piecewise affine, such that $1_{[t+\varepsilon_\ell,\infty[} \leqslant f_\ell \leqslant 1_{[t,\infty[}$. Thus, $\varepsilon_\ell f_\ell(s) = s - t$ for $s \in [t, t+\varepsilon_\ell]$. Define $R_\ell = \frac{1}{\varepsilon_\ell} \cdot T \llcorner (1_{\{t<\varphi<t+\varepsilon_\ell\}}\, d\varphi)$. Then

$$\partial T \llcorner (f_\ell \circ \varphi) - \partial(T \llcorner f_\ell \circ \varphi) = T \llcorner d(f_\ell \circ \varphi) = \frac{1}{\varepsilon_\ell} \cdot T \llcorner (1_{\{t<\varphi\leqslant t+\varepsilon_\ell\}}\, d\varphi) = R_\ell$$

by corollary 5.2.11 and locality. Since $d^2\omega = (1, 1, \omega_0, \dots)$, $\partial^2 = 0$ on currents. Thus,

$$\partial R_\ell = \partial(\partial T \llcorner (f_\ell \circ \varphi)) = -\partial T \llcorner d(f_\ell \circ \varphi) = -\frac{1}{\varepsilon_\ell}\partial T \llcorner (1_{\{t<\varphi\leqslant t+\varepsilon_\ell\}}),$$

by essentially the same argument as above. Let $K^j = \bigcup_{i \leqslant j} K_i$. Then

$$(\|R_\ell\| + \|\partial R_\ell\|)(X \setminus K^j) = \frac{1}{\varepsilon_\ell} \cdot \mu\big(\{t < \varphi \leqslant t + \varepsilon_\ell\} \setminus K^j\big) \leqslant \sum_{i>j} g_i'(t).$$

Hence, by theorem 5.4.2, extracting a subsequence, (R_ℓ) converges to some $R \in \mathbf{N}_{k-1}(X)$ such that $\|R\| + \|\partial R\|$ is concentrated on $\bigcup_j K_j$. Since $f_\ell \circ \varphi \to \varphi$ in $\mathbf{L}^1(\mu)$, we find $R = \partial T \llcorner \{\varphi > t\} - \partial(T \llcorner \{\varphi > t\})$, and the limit is independent of the subsequence chosen, so $R = \frac{\mathrm{d}}{\mathrm{d}t} S_t \llcorner d\varphi$. Moreover,

$$\|\partial(T \llcorner \{\varphi > t\})\|(\varphi^{-1}(t)) = \|R\|(\varphi^{-1}(t)) \leqslant \mathbf{M}(R) \leqslant \liminf_\ell \mathbf{M}(R_\ell) \leqslant f'(t)$$

since mass is l.s.c. with respect to pointwise convergence. □

5.4.6. The second lemma required is an improvement of the representation of mass in theorem 5.2.8. To that end, recall that for Borel measures μ, μ_j one says that μ is the supremum of $(\mu_j)_{j\in J}$, $\mu = \bigvee_{j\in J}\mu_j$, if for all $B \in \mathcal{B}(X)$,

$$\mu(B) = \sup\Big\{\sum_{j\in J}\mu_j(B_j)\ \Big|\ (B_j)_{j\in J}\text{ Borel partition of }B\Big\} .$$

Lemma 5.4.7. Let $A \subset X$ be σ-compact. There exists a countable $D \subset \mathrm{Lip}_b(X)\cap\mathrm{Lip}_1(X)$ which is dense in the sense that any bounded 1-Lipschitz function is the uniform limit on compact of a uniformly bounded sequence in D, such that

$$\|T\| = \bigvee\big\{\|T \llcorner\ d\pi\|\ \big|\ \pi_i \in D^k\big\}\quad\text{for all }\ T \in \mathbf{M}_k(X)\text{ concentrated on }A .$$

Proof. Let $A = \bigcup_j K_j$ with $K_j \subset X$ compact and write $X = \mathrm{Lip}_b(X)\cap\mathrm{Lip}_1(X)$. From theorem 5.2.8, the equation is valid for any current T, with X in place of D. For each $j \in \mathbb{N}$, we may choose a countable subset $D_j \subset X$ such that $f \in X$ is the uniform limit on K_j of a bounded sequence in D_j. Set $D = \bigcup_j D_j$ and fix $T \in \mathbf{M}_k(X)$ concentrated on A. Then it suffices to prove that

$$\|T \llcorner\ d\pi\| \llcorner\ K_j \leqslant \mu_j = \bigvee\big\{\|T \llcorner\ d\tau\|\ \big|\ \tau_i \in D_j\big\}\quad\text{for all }\ \pi \in X^k .$$

Let $f \in \mathcal{B}_b(X)$ vanish outside K_j, and let $(\pi_i^m) \subset D_j)$ be uniformly bounded, such that $\pi_i^m \to \pi_i$ uniformly on K_j. Let

$$\tau_i^m(x) = \inf_{y\in K_j}\big(\pi_i^m(y) + d(x,y)\big)\quad\text{and}\quad \tau_i = \int_{y\in K_j}\big(\pi_i(y)+d(x,y)\big) .$$

Then $\tau_1, \tau_i^m \in \mathrm{Lip}_1(X)$, $\tau_i^m = \pi_i^m$ on K_j, and $\tau_i^m \to \tau_i$ pointwise. By locality and continuity,

$$\langle f\,d\pi : T\rangle = \langle f\,d\tau : T\rangle = \lim_m\langle f\,d\tau^m : T\rangle = \lim_m\langle f\,d\pi^m : T\rangle \leqslant \int|f|\,d\mu_j .$$

This implies our claim and thus, the lemma. ☐

Proof of theorem 5.4.4. First, let $m = 1$, and set $S_x = \partial T \llcorner\ \{\pi > x\} - \partial(T \llcorner\ \{\pi > x\})$. By lemma 5.4.5, we have $S_x = \dfrac{\mathrm{d}}{\mathrm{d}x}T \llcorner\ (1_{\{\pi>x\}}\,d\pi)$ for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}$. This implies that S_x is concentrated on $L \cap \pi^{-1}(x)$, and moreover, for all $\omega \in \mathcal{D}^p(X)$, $0 \leqslant p < k$,

$$\mathbf{M}(S_x \llcorner\ \omega) \leqslant \frac{\mathrm{d}}{\mathrm{d}x}\|T \llcorner\ (\omega\wedge d\pi)\|(\{\pi \leqslant x\})\quad\text{for }\mathcal{L}^1\text{ a.e.}\quad x \in\in \mathbb{R} .$$

Hence, by theorem 3.2.1,

$$\int\mathbf{M}(S_x \llcorner\ \omega)\,dx \leqslant \int\frac{\mathrm{d}}{\mathrm{d}x}\|T \llcorner\ (\omega\wedge d\pi)\|(\{\pi \leqslant x\})\,dx \leqslant \mathbf{M}(T \llcorner\ (\omega\wedge d\pi)) . \quad (***)$$

We now check $(**)$ in this case. So, let φ be continuous of compact support, and set

$\gamma(t) = \int_{-\infty}^{t} \varphi \, d\mathcal{L}^1$. Note

$$\int_{\mathbb{R}} \varphi(x) 1_{\{\pi > x\}}(y) \, dx = \int_{-\infty}^{\pi(y)} \varphi(x) \, dx = \gamma(\pi(y)) \, .$$

Thus, for $f \, d\tau \in \mathcal{D}^{k-1}(X)$, we obtain (since $\gamma' = \varphi$)

$$\begin{aligned}
\int_{\mathbb{R}} \langle f \, d\tau : S_x \rangle \varphi(x) \, dx &= \int_{\mathbb{R}} \langle 1_{\{\pi > x\}} \cdot f \, d\tau : \partial T \rangle \varphi(x) \, dx - \int_{\mathbb{R}} \langle 1_{\{\pi > x\}} \, df \wedge d\tau : T \rangle \varphi(x) \, dx \\
&= \langle \gamma \circ \pi \cdot f \, d\tau : \partial T \rangle - \langle \gamma \circ \pi \, df \wedge d\tau : T \rangle \\
&= \langle f \, d(\gamma \circ \pi) \wedge d\tau : T \rangle = \langle f \cdot (\varphi \circ \pi) \, d\pi \wedge d\tau : T \rangle \\
&= \langle f \, d\tau : T \llcorner \, (\varphi \circ \pi) \, d\pi \rangle \, ,
\end{aligned}$$

which is just $(**)$. In particular,

$$|\langle f \, d\tau : T \llcorner \, d\pi \rangle| = \left| \int \langle f \, d\tau : S_x \rangle \, dx \right| \leqslant \prod_{j=1}^{k-1} \mathrm{Lip}(\tau_j) \cdot \iint |f| \, d\|S_x\| \, dx \, .$$

Hence, $\|T \llcorner \, d\pi\| \leqslant \int \|S_x\| \, dx$. Together with $(***)$, this implies the weak measurability of $x \mapsto \|S_x\|$ and $(*)$. This completes the proof of existence in case $m = 1$.

Assuming the existence statement is true for some $1 \leqslant m \leqslant k-1$, we intend to deduce it for $m+1$. To that end, write $\pi = (\pi_1, \bar{\pi})$ with $\pi_1 \in \mathrm{Lip}(X)$, and $x = (t, y)$. The statement being true for m and for 1 , we may define $T_t = \langle T, \pi_1, t \rangle$ and $T_x = \langle T_t, \bar{\pi}, y \rangle$. By assumption, we have

$$\iint \|T_x\| \, dy \, dt = \int \|T_t \llcorner \, d\bar{\pi}\| \, dt = \|T_t \llcorner \, (d\pi_1 \wedge d\bar{\pi})\| = \|T \llcorner \, d\pi\| \, .$$

The $\mathcal{L}^{m+1}$-measurability follows from Fubini's theorem 1.6.2. Similarly, for $\varphi = \varphi_1 \otimes \bar{\varphi}$ where $\varphi_1 \in \mathcal{K}(\mathbb{R})$ and $\bar{\varphi} \in \mathcal{K}(\mathbb{R}^m)$,

$$\begin{aligned}
\int T_x \varphi(x) \, dx &= \iint T_{(t,y)} \bar{\varphi}(y) \, dy \varphi_1(t) \, dt = \int T_t \llcorner \, (\bar{\varphi} \circ \bar{\pi} \, d\bar{\pi}) \varphi_1(t) \, dt \\
&= T \llcorner \, \big((\varphi_1 \circ \pi_1) \cdot (\bar{\varphi} \circ \bar{\pi}) \, d\pi_1 \wedge d\bar{\pi} \big) = T \llcorner \, (\varphi \circ \pi \, d\pi) \, .
\end{aligned}$$

Since $\mathcal{K}(\mathbb{R}) \otimes \mathcal{K}(\mathbb{R}^m)$ is dense in $\mathcal{K}(\mathbb{R}^{m+1})$ for the topology induced by $\mathbf{L}^1(\mathbb{R}^m)$, the equation $(**)$ follows. Hence, we have proved existence.

As to uniqueness, fix $f \, d\tau \in \mathcal{D}^{k-m}(X)$. Let ϱ_ε be a Dirac sequence. Then $(**)$ gives

$$\langle f \, d\tau : T_x \rangle = \lim_{\varepsilon \to 0+} \int \langle f \, d\tau : T_y \rangle \varrho_\varepsilon(y - x) \, dy = \lim_{\varepsilon \to 0+} \langle f \varrho_\varepsilon(\pi - x) \, d\tau \wedge d\pi : T \rangle$$

for $\mathcal{L}^1$ a.e. $x \in \mathbb{R}^m$. Hence $\langle \omega : T_x \rangle$ is determined a.e. uniquely for fixed $\omega = f \, d\tau$. Now fix D as in lemma 5.4.7, applied to the σ-compact $A = L \cup L'$, and let $N \subset \mathbb{R}^m$ be an $\mathcal{L}^m$

negligible Borel set such that

$$\langle f\, d\tau : T_x \rangle = \langle f\, d\tau : \langle T, \pi, x \rangle \rangle \quad \text{for all } \ x \in \mathbb{R}^m \setminus N\,,\ f \in \mathrm{Lip}_b(X)\,,\ \tau \in D^k\,.$$

Now, $\|(T_x - \langle T, \pi, x\rangle) \llcorner d\tau\| = 0$ for all $\tau \in D^k$ implies $\|T_x - \langle T, \pi, x\rangle\| = 0$ by the lemma. Hence the assertion. □

5.4.8. Using approximation by normal currents, we extend the slicing operators to rectifiable currents.

Slicing Theorem for Rectifiable Currents 5.4.9. Let $T \in \mathcal{R}_k(X)$ (resp. $T \in \mathcal{I}_k(X)$) and $\pi \in \mathrm{Lip}(X, \mathbb{R}^m)$, $1 \leqslant m \leqslant k$. There exists $\langle T, \pi, x\rangle \in \mathcal{R}_{k-m}(X)$ (resp. $\langle T, \pi, x\rangle \in \mathcal{I}_{k-m}(X)$) concentrated on $S_T \cap \pi^{-1}(x)$, such that equations $(*)$ and $(**)$ from slicing theorem for rectifiable currents 5.4.4 hold,

$$\langle T \llcorner B, \pi, x\rangle = \langle T, \pi, x\rangle \llcorner B \quad \text{for all } \ B \in \mathcal{B}(X)\,,$$

and

$$\int \mathbf{S}(\langle T, \pi, x\rangle)\, dx \leqslant \frac{\omega_m \omega_{k-m}}{\omega_k} \cdot \prod_{j=1}^{m} \mathrm{Lip}(\pi_j) \cdot \mathbf{S}(T)\,.$$

Moreover, whenever $T_x \in \mathbf{M}_{k-m}(X)$ are concentrated on $L \cap \pi^{-1}(x)$ for some σ-compact $L \subset X$, satisfy $(**)$, and $x \mapsto \mathbf{M}(T_x)$ is $\mathcal{L}^m$ summable, then $T_x = \langle T, \pi, x\rangle$ for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$.

We require the following lemma.

Lemma 5.4.10. Let $A \subset X$ be σ-compact. There exists a countable family $\mathcal{U}$ of open subsets of X such that for all open $U \subset X$, there exists $(U_j) \subset U$ such that $1_U = \lim_j 1_{U_j}$ in $\mathbf{L}^1(\mu)$ for any finite Borel measure μ concentrated on A.

Proof. Write $A = \bigcup_j K_j$ with $K_j \subset X$ compact. Let D be as in lemma 5.4.7 and let $\mathcal{U} = \big\{\{\pi > \frac{1}{2}\} \mid \pi \in D\big\}$. Let $U \subset X$ be open. Then $1_U = \sup_j f_j$ where $f_{j+1} \geqslant f_j \geqslant 0$ are 1-Lipschitz functions. Let $g_j \in D$, $|f_j - g_j| \leqslant \frac{1}{j}$ on K_j. By Lebesgue's dominated convergence theorem 1.5.7,

$$1_U = \lim_j 1_{\{f_j > \frac{1}{2}\}} \quad \text{in} \quad \mathbf{L}^1(\mu)$$

whenever μ is concentrated on A. □

Proof of theorem 5.4.9. First, assume $X = E^*$ for E separable Banach space. Then theorem 5.3.5 shows that there exist $T_j \in \mathbf{N}_k(X)$, such that $T = \sum_{j=0}^{\infty} T_j$ in $\mathbf{M}_k(X)$. Thus

$$\int \sum_{j=0}^{\infty} \mathbf{M}(\langle T_j, \pi, x\rangle)\, d\mathcal{L}^m(x) \leqslant \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \cdot \sum_{j=0}^{\infty} \mathbf{M}(T_j) = \prod_{i=1}^{m} \mathrm{Lip}(\pi_i) \cdot \mathbf{M}(T) < \infty\,.$$

Thus, for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}^m$, $\sum_j \langle T_j, \pi, x\rangle$ converges in $\mathbf{M}_{k-m}(X)$ to some $\langle T, \pi, x\rangle$. Then $\|\langle T, \pi, x\rangle\| \ll \mathcal{H}^{k-m}$, $\langle T, \pi, x\rangle$ is concentrated on $S_T \cap \pi^{-1}(x)$, and equations $(*)$ and

$(**)$ hold. In particular, $\langle T, \pi, x\rangle$ is rectifiable. In the general case, T is concentrated on $\operatorname{supp} T$, which is separable by theorem 1.2.5. Hence, the latter can be embedded into $\ell^\infty = (\ell^1)'$, and we obtain the slices by applying the isometric embedding.

Note that $\langle T, \pi, x\rangle$ is concentrated on S_T for $\mathcal{L}^m$ a.e. $x \in \mathbb{R}$. Hence, from proposition 5.3.6 and theorem 2.4.5, we deduce

$$\int \mathbf{S}(\langle T, \pi, x\rangle)\, dx = \int \mathcal{H}^{k-m}(S_T \cap \pi^{-1}(x))\, dx \leqslant \frac{\omega_m \omega_{k-m}}{\omega_k} \cdot \prod_{j=1}^{m} \operatorname{Lip}(\pi_j) \mathbf{S}(T)\,,$$

which gives the size inequality and thus establishes existence in the rectifiable case. Uniqueness follows as in theorem 5.4.4. This implies, for fixed $B \in \mathcal{B}(X)$,

$$\langle T \llcorner B, \pi, x\rangle = \langle T, \pi, x\rangle \llcorner B \quad \text{for } \mathcal{L}^m \text{ a.e. } \quad x \in \mathbb{R}^m\,.$$

Let $\mathcal{U}$ be as in lemma 5.4.10, applied to S_T. There exists an $\mathcal{L}^m$ Borel set $N \subset \mathbb{R}^m$ such that $\langle T \llcorner U, \pi, x\rangle = \langle T, \pi, x\rangle \llcorner U$ for all $x \in \mathbb{R}^n \setminus N$, $U \in \mathcal{U}$. Then the assertion follows for any open set, and then, by Borel regularity, for any Borel set.

Remains to treat the integer-rectifiable case. Suffices to consider $T = \varphi_\bullet [\![\vartheta]\!]$ for $\vartheta \in \mathbf{L}^1(\mathbb{R}^k, \mathbb{Z})$ of compact support. From the construction of slices in theorem 5.4.4, we find that $\langle [\![\vartheta]\!], \pi, x\rangle$ are integer-rectifiable, hence so are $T_x = \varphi_\bullet \langle [\![\vartheta]\!], \pi, x\rangle$. We apply the uniqueness statement to deduce $\langle T, \pi, x\rangle = T_x$ for $\mathcal{L}^m$ a.e. x. □

5.4.11. The following proposition on iterated slices and projections of slices follow from uniqueness.

Proposition 5.4.12. Let $T \in \mathbf{M}_k(X)$ be rectifiable or normal, $1 \leqslant m \leqslant k$, and fix a map $\pi \in \operatorname{Lip}(X, \mathbb{R}^m)$.

(i). For any $1 \leqslant n \leqslant k - m$ and any $\varphi \in \operatorname{Lip}(X, \mathbb{R}^n)$, we have

$$\langle T, (\pi, \varphi), (t, y)\rangle = \big\langle \langle T, \pi, t\rangle, \varphi, y\big\rangle \quad \text{for } \mathcal{L}^{m+n} \text{ a.e. } \quad (t, y) \in \mathbb{R}^{m+n}\,.$$

(ii). Let T be rectifiable. For any $n > m$ and $\varphi \in \operatorname{Lip}(X, \mathbb{R}^{n-m})$,

$$q_\bullet \langle (\varphi, \pi)_\bullet T, p, x\rangle = \varphi_\bullet \langle T, \pi, x\rangle \quad \text{for } \mathcal{L}^m \text{ a.e. } \quad x \in \mathbb{R}^m\,,$$

where $p : \mathbb{R}^n \to \mathbb{R}^m$ and $q = 1 - p : \mathbb{R}^n \to \mathbb{R}^{n-m}$ are given by

$$p(x) = (x_{n-m}, x_{n-m+1}, \dots, x_n) \quad \text{for all } \ x \in \mathbb{R}^n\,.$$

5.5 Closure and Boundary Rectifiability Theorems

5.5.1. As before, we assume that X is complete and contains a dense subset whose cardinality is an Ulam number. The following theorem characterises the (integer) rectifiability of normal currents through the (integer) rectifiability of their slices.

Theorem 5.5.2. Let $T \in \mathbf{N}_k(X)$. Then $T \in \mathcal{R}_k(X)$ if and only if for all $\pi \in \mathrm{Lip}(X, \mathbb{R}^k)$,

$$\langle T, \pi, x \rangle \in \mathcal{R}_0(X) \quad \text{for } \mathcal{L}^k \text{ a.e.} \quad x \in \mathbb{R}^k \, . \tag{$*$}$$

Similarly, $T \in \mathbf{I}_k(X)$ if and only if condition $(*)$ holds for $\mathbf{I}_0(X)$ (or $\mathcal{I}_0(X)$) in place of $\mathcal{R}_0(X)$.

5.5.3. Our strategy of proof is the following: Show that the slice map $x \mapsto \langle T, \pi, x \rangle$ is a *metric BV function* (to be defined). This implies that $\|T \llcorner d\pi\|$ is concentrated on a countably k-rectifiable set. A separability argument (in fact, lemma 5.4.7) then gives the corresponding statement for $\|T\|$, and this implies the rectifiability of T. To make this strategy rigorous, we need to establish the necessary facts on metric BV functions.

5.5.4. A metric space S is said to be *weakly separable* if there exists a countable family $\Phi \subset \mathrm{Lip}_b(S) \cap \mathrm{Lip}_1(S)$ such that

$$d(x,y) = \sup\nolimits_{\varphi \in \Phi} |\varphi(x) - \varphi(y)| \quad \text{for all } \ x, y \in S \, .$$

Fix a weakly separable metric space S, together with a separating family Φ. Then a function $f : \mathbb{R}^k \to S$ is said to be of *metric bounded variation*, if for all $\varphi \in \Phi$, $\varphi \circ f$ is on locally bounded variation, and

$$\|\nabla f\| = \bigvee\nolimits_{\varphi \in \Phi} |\nabla \varphi \circ f|$$

is a finite Borel measure. We denote the set of these functions by $\mathrm{MBV}(\mathbb{R}^k, S)$. We shall see presently that this definition is independent of the choice of Φ.

Proposition 5.5.5. Let $f \in \mathrm{MBV}(\mathbb{R}^k, S)$ and $\psi \in \mathrm{Lip}_b(S) \cap \mathrm{Lip}_1(S)$. Then $\psi \circ f$ is locally of bounded variation, and $|\nabla(\psi \circ f)| \leqslant \|\nabla f\|$. In particular,

$$\|\nabla f\| = \bigvee\nolimits_{\psi \in \mathrm{Lip}_b(S) \cap \mathrm{Lip}_1(S)} |\nabla(\psi \circ f)| \, .$$

Proof. First, let $k = 1$, $I \subset \mathbb{R}$ an open interval, and $g : I \to \mathbb{R}$ a bounded function. Let L_g be the Lebesgue set, i.e. the set of points where the precise representative of g exists and equals g, cf. theorem 1.9.9. Let $|\nabla g|(I)$ be the total variation of g if g is locally of bounded variation, and ∞ otherwise. Then

$$|\nabla g|(I) = \sup \left\{ \sum_{i=0}^{p-1} |g(t_{i+1}) - g(t_i)| \;\middle|\; t_0 < t_1 < \cdots < t_p \, , \, t_i \in I \setminus N \right\}$$

whenever N is a Lebesgue zero set containing the complement of L_g. (This is the so-called essential total variation.) This follows by similar arguments as in the proof of theorem 3.2.1.

In particular, the set of point masses of $|\nabla(\varphi \circ f)|$ is $\mathcal{L}^1$ negligible for any $\varphi \in \Phi$. Let

$$N = I \setminus L_{\psi\circ f} \cup \bigcup\nolimits_{\varphi\in\Phi} \big[I \setminus L_{\varphi\circ f} \cup \{ t \in I | |\nabla(\varphi \circ f)|(t) > 0 \} \big] .$$

Thus, because ψ is 1-Lipschitz,

$$|(\psi \circ f)(t) - (\psi \circ f)(s)| \leqslant d\big(f(t), f(s)\big) = \sup_{\varphi\in\Phi} |(\varphi \circ f)(t) - (\varphi \circ f)(s)| \leqslant \|\nabla f\|\big(]s, t[\big)$$

for all $s < t$, $s, t \in I \setminus N$. Thus, $|(\psi \circ f)|(I) \leqslant \|\nabla f\|(I)$. Standard approximation arguments using Borel regularity entail the claim in case $k = 1$.

Now, let $k > 1$. For any function $g : \mathbb{R}^k \to \mathbb{R}$ of locally bounded variation, and any $e \in \mathbb{S}^{k-1}$, $x \in e^\perp$, define

$$g_{e,x}(t) = g(x + te) \quad \text{and} \quad V_g^{e,x}(B) = |\nabla g_{e,x}| \big\{ t \in \mathbb{R} \bigm| x + te \in B \big\} \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^k) .$$

Moreover, let

$$|\nabla_e f|(B) = \int_{e^\perp} V_g^{e,x}(B) \, d\mathcal{H}^{k-1}(x) \quad \text{for all } B \in \mathcal{B}(\mathbb{R}^k) .$$

Then Fubini's theorem 1.6.2 and the invariance of $\mathcal{L}^k$ under translation and orthogonal transformations, which follows from $\mathcal{L}^k = \mathcal{H}^k$ and the corresponding statements for $\mathcal{H}^k$ (theorem 2.2.4), show that for all relatively compact open subsets $U \subset \mathbb{R}^k$ and all $\varphi \in \mathcal{C}_c^{(1)}(\mathbb{R})$, $|\varphi| \leqslant 1$,

$$\int_{e^\perp} \int_{(e:U_x)} g(x + te)\varphi'(t) \, dt \, d\mathcal{H}^{k-1}(x) = \int_U g(x)\varphi'\big((e : x)\big) \, \mathcal{L}^k(x) \leqslant |\nabla g|(U) ,$$

where $U_x = (x + \mathbb{R}e) \cap U$. By standard approximation arguments, we find

$$|\nabla g| = \bigvee\nolimits_{e\in\mathbb{S}^{k-1}} |\nabla_e g| .$$

By the case $k = 1$, we have $V_{\psi\circ f}^{e,x} \leqslant \bigvee_{\varphi\in\Phi} V_{\varphi\circ f}^{e,x}$ for all $e \in \mathbb{S}^{n-1}$, $x \in e^\perp$. For $e \in \mathbb{S}^{k-1}$,

$$\begin{aligned} |\nabla_e(\psi \circ f)| &= \int_{e^\perp} V_{\psi\circ f}^{e,x} \, d\mathcal{H}^{k-1}(x) = \int_{e^\perp} \bigvee\nolimits_{\varphi\in\Phi} V_{\varphi\circ f}^{e,x} \, d\mathcal{H}^{k-1}(x) \\ &= \bigvee\nolimits_{\varphi\in\Phi} \int_{e^\perp} V_{\varphi\circ f}^{e,x} \, d\mathcal{H}^{k-1}(x) = \bigvee\nolimits_{\varphi\in\Phi} |\nabla_e(\varphi \circ f)| \leqslant \|\nabla(\varphi \circ f)\| , \end{aligned}$$

where we have used σ-additivity and monotone convergence to exchange $\bigvee$ and the integral. Taking the $\bigvee_{e\in\mathbb{S}^{k-1}}$, the assertion follows. □

5.5.6. Consider the space $\mathrm{Lip}_b(X)$. It is a Banach space, endowed with the *flat norm*

$$\mathbf{F}(\varphi) = \|\varphi\|_\infty + \mathrm{Lip}(\varphi) \quad \text{for all } \varphi \in \mathrm{Lip}_b(X)\,.$$

Dually, the space of 0-dimensional currents $\mathbf{M}_0(X)$ can be endowed with the *flat norm*

$$\mathbf{F}(T) = \sup_{\varphi \in \mathrm{Lip}_b(X)\,,\,\mathbf{F}(\varphi)\leqslant 1} |\langle \varphi : T\rangle| \quad \text{for all } T \in \mathbf{M}_0(X)\,.$$

With this norm, $\mathbf{M}_0(X)$ is weakly separable whenever X is weakly separable.

In fact, if X is weakly, separable, then $\mathbf{M}_0(X)$ is isometrically embedded into $\mathbf{M}_0(\ell^\infty)$. Furthermore,

$$\bigcup_{n\in\mathbb{N}} \{\varphi \in \mathrm{Lip}_b(\mathbb{R}^n) \mid \mathbf{F}(\varphi) \leqslant 1\}$$

is a norm-determining subset and separable subset of the unit ball $\mathrm{Lip}_b(\ell^\infty)$.

Rectifiability for MBV 5.5.7. Let X be weakly separable, let $S = \mathbf{M}_0(X)$, endowed with the flat norm, and let $T \in \mathrm{MBV}(\mathbb{R}^k, S)$. Then there exists $N \subset \mathbb{R}^k$, $\mathcal{L}^k(N) = 0$, such that for all compact $K \subset X$,

$$R_K = \bigcup_{x\in\mathbb{R}^k\setminus N} \{y \in K \mid \|T_x\|(y) > 0\}$$

is contained in a countably k-rectifiable subset of X.

5.5.8. The proof requires a version of the mean value inequality. To that end, for any real-valued function F defined on the bounded subsets of X, let

$$M(F)(x) = \sup_{r>0} \frac{F\big(B(x,r)\big)}{\omega_k r^k} \quad \text{for all } x \in X$$

define the *maximal function* of F.

Lemma 5.5.9. Let S be weakly separable and $f \in \mathrm{MBV}(\mathbb{R}^k, S)$. Define

$$c_k = \frac{\mathcal{L}^k\big(B(x,\|x-y\|) \cap B(y,\|x-y\|)\big)}{\|x-y\|^k} \quad \text{for all } x,y \in \mathbb{R}^k\,,\ x \neq y\,,$$

a constant depending only on k. Then there exists $N \subset \mathbb{R}^k$, $\mathcal{L}^k(N) = 0$, such that

$$d\big(f(x), f(y)\big) \leqslant \frac{\omega_k}{c_k}\big(M(\|\nabla f\|)(x) + M(\|\nabla f\|)(y)\big) \cdot \|x-y\| \quad \text{for all } x,y \in \mathbb{R}^k \setminus N\,.$$

Proof. Let $g : \mathbb{R}^k \to \mathbb{R}$ be locally of bounded variation and write L_g for its set of Lebesgue points, cf. corollary 1.9.12. By apply the change of variable formula (corollary 4.1.6) to the bi-Lipschitz function $[0,1] \times \mathbb{S}^{k-1} \to B(x,r) : (t,e) \mapsto x + tre$, we obtain

$$\frac{1}{\omega_k r^k} \int_{B(x,r)} \frac{|g(x) - g(y)|}{\|x-y\|}\, d\mathcal{L}^k(y) = \frac{1}{\omega_k r^k} \int_0^1 \int_{\mathbb{S}^{k-1}} t^{1-k} \cdot \frac{g(x) - g(x+rte)}{tr} d\,\mathcal{H}^{k-1}(e)\, dt$$

$$\leqslant \int_0^1 \frac{|\nabla g|(B(x,tr))}{\omega_k (tr)^k}\, dt \leqslant M(|\nabla g|)(x)$$

for every $x \in L_g$. Let $x, y \in L_g$, $x \neq y$, and set $r = \|x - y\|$. Then

$$\begin{aligned}\frac{|g(x)-g(y)|}{r}\,\frac{1}{r^k c_k}\cdot \int_{B(x,r)\cap B(y,r)} &\frac{|g(x)-g(y)|}{r}\, d\mathcal{L}^k(z) \\ &\leqslant \frac{1}{r^k c_k}\cdot \int_{B(x,r)\cap B(y,r)} \left[\frac{|g(x)-g(z)|}{\|x-z\|} + \frac{|g(z)-g(y)|}{\|z-y\|}\right] d\mathcal{L}^n(z)\,.\end{aligned}$$

because $\|x - z\|, \|y - z\| \leqslant r$. We obtain

$$|g(x) - g(y)| \leqslant \frac{\omega_k}{c_k} \cdot \left[M(|\nabla g|)(x) + M(|\nabla g|)(y)\right] \cdot \|x - y\|\,.$$

The assertion follows by letting $g = \varphi \circ f$, $\varphi \in \Phi$, where Φ is any countable separating family for S , and defining $N = \bigcup_{\varphi\in\Phi} \mathbb{R}^k \setminus L_{\varphi\circ f}$. □

Proof of theorem 5.5.7. Apply lemma 5.5.9 with $S = \mathbf{M}_0(X)$ to obtain a negligible set $M \subset \mathbb{R}^k$ and define $N = M \cup \{M(\|\nabla T\|) = \infty\}$. Let $K \subset X$ be compact, fix $\varepsilon, \delta > 0$, and let

$$A_{\varepsilon\delta} = \left\{M(\|\nabla T\|) < \tfrac{1}{2\varepsilon}\right\} \cap \bigcap\nolimits_{y\in K} \left\{x \in \mathbb{R}^k \setminus N \;\middle|\; \|T_x\|(y) \geqslant \varepsilon \;\Rightarrow\; \|T_x\|(B(y, 3\delta) \setminus y) \leqslant \tfrac{\varepsilon}{3}\right\}$$

and

$$R_{\varepsilon\delta} = \left\{x \in K \;\middle|\; \exists y \in A_{\varepsilon\delta} \,:\, \|T_x\|(y) \geqslant \varepsilon\right\}.$$

Then $R_K = \bigcup_{\varepsilon,\delta>0} R_{\varepsilon\delta}$, and since it suffices to consider a countable subfamily of the sets $R_{\varepsilon\delta}$, it suffices to prove that these are contained in countably k-rectifiable subsets of X .

So let $B \subset R_{\varepsilon\delta}$ be of diameter $\leqslant \delta$. For $x_1, x_2 \in B$, and $y_1, y_2 \in A_{\varepsilon\delta}$, $\|T_{x_i}\|(y_i) \geqslant \varepsilon$, we claim

$$d(x_1, x_2) \leqslant \frac{3c_k(\delta + 1)}{\varepsilon^3} \cdot \|y_1 - y_2\|\,.$$

In fact, let $d = d(x_1, x_2)$. Let $\phi \in \mathrm{Lip}_1(X)$, $1 \leqslant \phi \leqslant d$, such that $\phi = d(\llcorner\!\lrcorner, x_1)$ on $B(x_1, d)$, $\phi = 0$ outside $B(x_1, 2\delta)$. Then

$$|\langle\phi : T_{y_1}\rangle| \leqslant \int_{B(2\delta)\setminus x} |\phi|\, d\|T_{y_1}\| \leqslant \frac{\varepsilon d}{3}\,,$$

and

$$|\langle\phi : T_{y_2}\rangle| \geqslant |\langle d : T_{y_2}\rangle| - |\langle d - \phi : T_{y_2}\rangle| \geqslant \varepsilon d - \frac{\varepsilon d}{3}\,.$$

Hence,

$$d(x_1, x_2) \leqslant \frac{3}{\varepsilon} \cdot \left|\langle\phi : T_{y_1}\rangle - \langle\phi : T_{y_2}\rangle\right| \leqslant \frac{3(d+1)}{\varepsilon} \cdot \mathbf{F}\left(T_{y_1} - T_{y_2}\right) \leqslant \frac{3c_k(d+1)}{\omega_k \varepsilon^3} \cdot \|y_1 - y_2\|\,,$$

by the inequality from the lemma and because the maximal function is less than $\frac{1}{2\varepsilon}$. This proves the claim.

By our assumptions on $A_{\varepsilon\delta}$, the map $f : y \mapsto x$ is well-defined on a subset of $A_{\varepsilon\delta}$, and surjective onto B . By our claim, it is Lipschitz. Since X is weakly separable, f can be extended to a Lipschitz function $\mathbb{R}^k \to \ell^\infty$ containing B in its image. Hence, B is contained in a countably k-rectifiable subset, and by weak separability, we find that this also true for $R_{\varepsilon\delta}$. □

Now we can prove our theorem on the rectifiability of normal currents.

Proof of theorem 5.5.2. Let $T \in \mathbf{N}_k(X)$. Then T is concentrated on a separable subspace, so by locality, we may assume that X is weakly separable. Let $S = \mathbf{M}_0(X)$, endowed with the flat norm. We contend that $x \mapsto T_x = \langle T, \pi, x\rangle$ is contained in $\mathrm{MBV}(\mathbb{R}^k, S)$. It is sufficient to do this under the assumption $\pi_i \in \mathrm{Lip}_1(X)$.

Let $\psi \in \mathcal{C}_c^{(1)}(\mathbb{R}^k)$ and $\varphi \in \mathrm{Lip}_b(X)$, $\mathbf{F}(\varphi) \leqslant 1$. By the defining equation for the slices, the chain rule, and corollary 5.2.11,

$$
\begin{aligned}
(-1)^{i-1}\int \langle \varphi : T_x\rangle \partial_i\psi(x)\, d\mathcal{L}^k(x) &= (-1)^{i-1}\langle \varphi \cdot \partial_i\psi\circ\pi : T \llcorner\, d\pi\rangle \\
&= \langle \varphi\, d(\psi\circ\pi) \wedge d\hat{\pi}_i : T\rangle = \langle \varphi\cdot\psi\circ\pi : \partial T\rangle - \langle \psi\circ\pi\, d\varphi \wedge d\hat{\pi}_i : T\rangle \\
&\leqslant \int |\psi\circ\pi|\, d\big(\|T\| + \|\partial T\|\big)
\end{aligned}
$$

where $d\hat{\pi}_i = d\pi_1 \wedge \cdots \wedge d\pi_{i-1} \wedge d\pi_{i+1} \wedge \cdots \wedge d\pi_k$. Hence, $x \mapsto \langle \varphi : T_x\rangle$ is locally of bounded variation, and $x \mapsto T_x$ lies in $\mathrm{MBV}(\mathbb{R}^k, S)$ with

$$
\|\nabla T_x\| \leqslant k \cdot \pi\big(\|T\| + \|\partial T\|\big) .
$$

Now for the rectifiable case. By theorem 5.4.9, the condition stated in the theorem is necessary. Conversely, let $L \subset X$ be a σ-compact on which T and ∂T are concentrated. Fix $\pi \in \mathrm{Lip}(X, \mathbb{R}^k)$. By theorem 5.5.7, there exists a negligible $N_\pi \subset \mathbb{R}^k$ and a countably k-rectifiable $R_\pi \subset X$ containing

$$
\bigcup\nolimits_{x\in\mathbb{R}^k\setminus N_\pi} \{y \in L \mid \|\langle T, \pi, x\rangle\|(y) > 0\} .
$$

Then

$$
\|T \llcorner\, d\pi\|(X \setminus R_\pi) = \|T \llcorner\, d\pi\|(L \setminus R_\pi) = \int_{\mathbb{R}^k} \|\langle T, \pi, x\rangle\|(L \setminus R_\pi)\, dx = 0 ,
$$

since, as follows from normality, $\|\langle T, \pi, x\rangle\| \ll \mathcal{H}^0$. Hence, $T \llcorner\, d\pi$ is concentrated on R_π . By lemma 5.4.7, T is concentrated on a countably k-rectifiable subset. Since $\|T\| \ll \mathcal{H}^k$ by normality, we have $T \in \mathcal{R}_k(X)$.

Necessity in the integer-rectifiable case also follows from theorem 5.4.9. Conversely, for any open $U \subset X$ and any $\pi \in \operatorname{Lip}(X, \mathbb{R}^k)$, we have

$$\langle \pi_\bullet(T \llcorner U), \mathrm{id}, x\rangle = \pi_\bullet \langle T \llcorner U, \varphi, x\rangle = \pi_\bullet\big(\langle T, \varphi, x\rangle\big) \in \mathcal{I}_0(\mathbb{R}^k) .$$

But the slices of $\pi_\bullet(T \llcorner U) = [\![\vartheta]\!] \in \mathbf{N}_k(\mathbb{R}^k)$ with respect to id are just $\vartheta(x)\delta_x$. Hence the integer-rectifiability of its slices immediately implies $\pi_\bullet(T \llcorner U) \in \mathcal{I}_k(\mathbb{R}^k)$, by theorem 5.3.3 (i). Thus, $T \in \mathcal{I}_k(X)$, and by normality, $T \in \mathbf{I}_k(X)$. □

We come the formulation of the all-important closure theorem.

Closure Theorem 5.5.10. Let $T, (T_j) \subset \mathbf{N}_k(X)$ such that $T = \lim_j T_j$ pointwise. If $T_j \in \mathcal{R}_k(X)$ and

$$\sup_j \mathbf{N}(T_j) + \mathbf{S}(T_j) < \infty ,$$

then $T \in \mathcal{R}_k(X)$. If $T_j \in \mathbf{I}_k(X)$ and is bounded in $\mathbf{N}_k(X)$, then $T \in \mathbf{I}_k(X)$.

We need the following lemma in the theorem's proof.

Lemma 5.5.11. Let $T, T_j \in \mathbf{N}_k(X)$ such that (T_j) is bounded and converges pointwise to T and fix $\pi \in \operatorname{Lip}(X)$. Then, for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, there exists a subsequence $\alpha = \alpha_t$, such that $(\langle T_{\alpha(j)}, \pi, t\rangle)$ is bounded in $\mathbf{N}_{k-1}(X)$ converges pointwise to $\langle T, \pi, t\rangle$. Moreover, if the T_j are rectifiable and $(\mathbf{S}(T_j))$ is bounded, then α can be chosen such that $(\mathbf{S}(\langle T_{\alpha(j)}, \pi, t\rangle))$ is bounded.

Proof. First prove the existence of a subsequence α such that $\langle T_{\alpha(j)}, \pi, t\rangle \to \langle T, \pi, t\rangle$ pointwise for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$. Recall

$$\langle T, \pi, t\rangle = \partial T \llcorner \{\pi > t\} - \partial\big(T \llcorner \{\pi > t\}\big) ,$$

so it suffices to prove the convergence of $T_{\alpha(j)} \llcorner \{\pi > t\}$ and $\partial T_{\alpha(j)} \llcorner \{\pi > t\}$. Define $\mu_j = \pi_\bullet(\|T_j\| + \|\partial T_j\|)$ and choose α such that $\mu_{\alpha(n)}$ is weak* convergent in $\mathcal{C}_c(\mathbb{R}^k)'$ to some finite Borel measure μ. If $t \in \mathbb{R}$ is not an atom of μ, then

$$\lim_{\delta\to 0+} \limsup_j \mu_{\alpha(n)}([t-\delta, t+\delta]) \leqslant \lim_{\delta\to 0+} \mu([t-\delta, t+\delta]) = 0 .$$

Define $\chi_\delta(x) = \max(0, \min(1, \delta^{-1}(\pi(x) - t)))$ for all $x \in X$. Then χ_δ is Lipschitz, and for $f \in \operatorname{Lip}_b(X)$, $\tau_i \in \operatorname{Lip}_1(X)$,

$$\big|\langle f\, d\tau : T_{\alpha(j)} \llcorner \chi_\delta - T_{\alpha(j)} \llcorner \{\pi > t\}\rangle\big| \leqslant \int_{\pi^{-1}[t,t+\delta]} |f|\, d\|T_{\alpha(j)}\| ,$$

and we obtain $T_{\alpha(j)} \llcorner \{\pi > t\} \to T \llcorner \pi > t$ pointwise. The same argument works for $\partial T_{\alpha(j)}$. Furthermore, by Fatou's lemma (theorem 1.5.5)

$$\begin{aligned}\int_{\mathbb{R}} \liminf_j \mathbf{N}(\langle T_{\alpha(j)}, \pi, t\rangle)\, dt &\leqslant \liminf_j \int_{\mathbb{R}} \mathbf{N}(\langle T_{\alpha(j)}, \pi, x\rangle)\, dt \\ &= \liminf_j \mathbf{N}(T_{\alpha(j)} \llcorner d\pi) \leqslant \operatorname{Lip}(\pi) \cdot \sup_j \mathbf{N}(T_j) < \infty .\end{aligned}$$

Hence, for a.e. $t \in \mathbb{R}$, there exists a subsequence $\beta = \beta_t$ of α such that $(\mathbf{N}(\langle T_{\beta(j)}, \pi, t\rangle))$ is bounded. Finally, for bounded sizes, as for the norms, we use the size estimate in theorem 5.4.9 to obtain subsequences for which the slices' sizes are bounded. □

Proof of theorem 5.5.10. The proof is by induction on k. First, let $k = 0$. In the rectifiable case, let $A \subset \operatorname{supp} T$ be any finite set. Then there is $\varepsilon > 0$ such that that the balls $B(a, \varepsilon)$, $a \in A$, are disjoint. By the lower semi-continuity of mass, for large j, each of the $B(a, \varepsilon)$ intersects $\operatorname{supp} T_j$, i.e. $T_j = \sum_{a \in A_j} \alpha_{aj} \delta_a$, and for all $a \in A$, there exists $a' \in A_j$ such that $d(a, a') \leqslant \varepsilon$. By the bounedness of size, $\# A_j$ is bounded, and it follows that T is a finite sum of point masses. Thus, $T \in \mathcal{R}_0(X)$.

In the integer-rectifiable case the bound on size follows from the bound on mass, since for $T = \sum_j \alpha_j \delta_{x_j}$, $\|T\| = \sum_j |\alpha_j| \delta_{x_j}$. Thus, the limit current is rectifiable, any we need only to see that its coefficients are integral, we is immediate from pointwise convergence.

Back in the rectifiable case, let $k \geqslant 1$ and assume the statement is true for $k-1$. We check that the assumptions of theorem 5.5.2 are verified. Write $\pi = (\pi_1, \bar{\pi})$, and set

$$S = T \llcorner \, d\pi_1 \,, \; S_j = T_j \llcorner \, d\pi_1 \,, \; S_t = \langle T, \pi_1, t\rangle \,, \; S_{jt} = \langle T_j, \pi_1, t\rangle \,.$$

From lemma 5.5.11, we find that for $\mathcal{L}^1$ a.e. $t \in \mathbb{R}$, S_t is the limit of a subsequence of S_{jt} with bounded sizes and bounded norms. By the inductive hypothesis, $S_t \in \mathcal{R}_{k-1}(X)$ for a.e. t. For any such t, $\langle T, \pi, (t,y)\rangle = \langle S_t, \bar{\pi}, y\rangle \in \mathcal{R}_0(X)$ for $\mathcal{L}^{k-1}$ a.e. $y \in \mathbb{R}^{k-1}$. Hence $\langle T, \pi, x\rangle \in \mathcal{R}_0(X)$ for $\mathcal{L}^k$ a.e. $x \in \mathbb{R}^k$. By theorem 5.5.2, this implies $T \in \mathcal{R}_0(X)$. The integer-rectifiable case follows in the same manner. □

Boundary Rectifiability Theorem 5.5.12. Let $T \in \mathbf{I}_k(X)$. Then $\partial T \in \mathbf{I}_{k-1}(X)$.

Proof. Since $\partial T = 0$ for $k = 0$, only the case $k \geqslant 1$ is interesting. Moreover, normality of ∂T is clear since $\partial^2 T = 0$, so we only need to establish integer-rectifiability. We proceed by induction on k, and begin with $k = 1$. Let $U \subset X$ be open and set $\varphi = \operatorname{dist}(\sqcup, X \setminus U)$. Then

$$\begin{aligned}\langle 1_{\{\varphi > t\}} : \partial T\rangle &= \langle 1 : \partial T \llcorner \, \{\varphi > t\}\rangle \\ &= \langle 1 : \partial (T \llcorner \, \{\varphi > t\})\rangle + \big\langle 1 : \langle T, \varphi, t\rangle\big\rangle = \big\langle 1 : \langle T, \varphi, 1\rangle\big\rangle \in \mathbb{Z}\end{aligned}$$

for $\mathcal{L}^1$ a.e. $t > 0$, by locality, theorem 5.3.3 (i), theorem 5.4.4 (iii), and theorem 5.4.9. By continuity of ∂T, we obtain $\langle 1_U : \partial T\rangle = \langle 1_{\varphi > 0} : \partial T\rangle \in \mathbb{Z}$. By theorem 5.3.3 (i), $\partial \in \mathcal{I}_0(X)$.

Assuming the statement for k, we prove it for $k+1$. Indeed, suffices to prove $\langle \partial T, \pi, x\rangle$ integer-rectifiable for $\pi \in \operatorname{Lip}(X, \mathbb{R}^k)$, by theorem 5.5.2. Write $\pi = (\pi_1, \bar{\pi})$. Then $\langle \partial T, \pi_1, t\rangle = -\partial \langle T, \pi_1, t\rangle$ is generically integer-rectifiable by the inductive hypothesis and the assumption on T. Then so is $\langle \partial T, \pi, (t,y)\rangle = \langle \langle \partial T, \pi_1, t\rangle, \bar{\pi}, y\rangle$. Hence follows the assertion. □

Besides the central closure and boundary rectifiability theorems, we briefly digress to give the following non-trivial characterisations of rectifiability as further corollaries to the rectifiability of slices theorem 5.5.2.

Theorem 5.5.13. Let $T \in \mathbf{N}_k(X)$. Then $T \in \mathcal{R}_k(X)$ if and only if it is concentrated on a Borel set σ-finite for $\mathcal{H}^k$.

Proof. The necessity is clear. Let S be an $\mathcal{H}^k$ σ-finite set on which T is concentrated. Let $\pi \in \operatorname{Lip}(X, \mathbb{R}^k)$. Since T is concentrated on a separable set, we find by corollary 2.2.3 that for $\mathcal{L}^k$ a.e. $y \in \mathbb{R}^k$, $S \cap \pi^{-1}(y)$ is countable. Since the slices $\langle T, \pi, x\rangle$ are concentrated on $S \cap \pi^{-1}(x)$, we find $\langle T, \pi, x\rangle \in \mathcal{R}_0(X)$, and theorem 5.5.2 gives the desired conclusion. □

Theorem 5.5.14. Let $T \in \mathbf{N}_k(X)$. Then the following holds.

(i). We have $T \in \mathbf{I}_k(X)$ if and only if $\varphi_\bullet T \in \mathcal{I}_k(\mathbb{R}^{k+1})$ for all $\varphi \in \operatorname{Lip}(X, \mathbb{R}^{k+1})$.

(ii). We have $T \in \mathbf{I}_k(X)$ if and only if $\pi_\bullet(T \llcorner B) \in \mathcal{I}_k(\mathbb{R}^k)$ for all $\pi \in \operatorname{Lip}(X, \mathbb{R}^k)$ and $B \in \mathcal{B}(X)$.

(iii). If $X = \mathbb{R}^n$ with any norm, then $T \in \mathcal{R}_k(X)$ if and only if $\varphi_\bullet T \in \mathcal{R}_k(\mathbb{R}^{k+1})$ for all $\varphi \in \operatorname{Lip}(X, \mathbb{R}^{k+1})$.

Proof. All the conditions are necessary. In every case, it suffices to prove $T_x = \langle T, \pi, x\rangle$ generically (integer-) rectifiable for any fixed $\pi \in \operatorname{Lip}(X)\mathbb{R}^k$, by theorem 5.5.2. Let $\mathcal{U}$ be the countable family of open sets from lemma 5.4.10 and denote $\varphi_U = \operatorname{dist}(\sqcup, X \setminus U)$ for $U \in \mathcal{U}$.

In the situation of (i), let $q : \mathbb{R}^{k+1} \to \mathbb{R}$ be the projection onto the first component, and $p : \mathbb{R}^{k+1} \to \mathbb{R}^k$ the projection onto the last k components. Applying proposition 5.4.12, we obtain an $\mathcal{L}^k$-negligible $N \subset \mathbb{R}^k$ such that for all $x \in \mathbb{R}^k \setminus N$ and $U \in \mathcal{U}$,

$$\varphi_{U\bullet} T_x = q_\bullet \langle (\varphi_U, \pi)_\bullet T, p, x\rangle \in \mathcal{I}_0(\mathbb{R}) .$$

In particular, $\langle 1_U : T_x\rangle = \langle 1_{]0,\infty[} : \varphi_{U\bullet} T\rangle \in \mathbb{Z}$ for all $x \in \mathbb{R}^k \setminus N$ and $U \in \mathcal{U}$. The same is true for $B \in \mathcal{B}(X)$ in place of U. Thus follows (i). Statements (ii) and (iii) are similar, where for (iii) one proceeds as in the proof of theorem 5.3.3 (iii). □

Remark 5.5.15. To this point, the presentation follows [AK00a] closely. We also mention that [Amb94, proof of th. 2.3] was essential for the comprehension of [AK00a, lem.7.3, th. 7.4], which are lemma 5.5.9 and theorem 5.5.7 in our enumeration.

5.6 Generalised Plateau's Problem

5.6.1. As usual, we assume that X is metric, complete, and contains a dense subset of Ulam cardinality. The following theorem, the solution of DeGiorgi's generalised Plateau's problem, is the culmination of our study of currents.

Existence of Minimising Currents 5.6.2. Let X be a Hilbert space and $S \in \mathbf{I}_k(X)$, such that $\partial S = 0$ and $\operatorname{supp} S$ is compact. Then there exists $T \in \mathbf{I}_k + 1(X)$ such that $\partial T = S$, T has compact support, and

$$\mathbf{M}(T) = \inf\{\mathbf{M}(R) \mid R \in \mathbf{I}_{k+1}(X), \partial R = S\} .$$

Remark 5.6.3. Theorem 5.6.2 is not the most general setting in which Plateau's problem can be solved. Ambrosio and Kirchheim [AK00a, th. 10.6] solve it for so-called *isoperimetric* Banach spaces $Y = E^*$ where E is separable. However, their proof requires the use of Gromov's compactness theorem. Wenger [Wen04] circumvents the use of this theorem, thereby showing that any Banach space is isoperimetric. He then proceed to extend the solution of Plateau's problem to *all* dual Banach spaces (not necessarily weak* separable) and all Hadamard spaces (i.e. non-positively curved and geodesically complete spaces).

5.6.4. The essential ingredients in the theorem's proof, the compactness of normal currents and the closure theorem, have already been assembled. The sole remaining step is the existence of solutions of the equation $\partial T = S$ with some control on their mass, commonly refered to as the existence of filling currents.

Cone Construction 5.6.5. Let $T \in \mathbf{N}_k(X)$, and define, for $t \in [0,1]$, metric functionals $t \times T$ on $[0,1] \times X$ by

$$\langle f\, d\pi : t \times T\rangle = \langle f_t\, d\pi_t : T\rangle \quad \text{where} \quad f_t = f(t, \sqcup) \quad \text{and} \quad d\pi_t = d\pi_{1t} \wedge \cdots \wedge d\pi_{kt} .$$

Furthermore, define the *cone* of T to be the metric function $[0,1] \times T$ given by

$$\langle f\, d\pi : T \times [0,1]\rangle = \sum_{j=1}^{k+1} (-1)^{j+1} \int_0^1 \left\langle f_t \frac{\partial \pi_{jt}}{\partial t}\, d\hat{\pi}_{it} : T \right\rangle dt .$$

If $\operatorname{supp} T$ is bounded, then $[0,1] \times T \in \mathbf{N}_{k+1}(X)$, $t \times T \in \mathbf{N}_k(X)$,

$$\|[0,1] \times T\| \leqslant (\mathcal{L}^1 \llcorner [0,1]) \otimes \|T\| ,$$

and

$$\partial\big([0,1] \times T\big) = 1 \times T - 0 \times T - [0,1] \times \partial T .$$

Moreover, if T is integral, then so is $[0,1] \times T$.

Proof. First, we prove that $[0,1] \times T$ is normal. Clearly, it is a multilineare metric functional. By

$$|\langle f\, d\pi : [0,1] \times T\rangle| \leqslant \sum_{j=1}^{k+1} \int_0^1 \left| \left\langle f_t \frac{\partial \pi_{jt}}{\partial t}\, d\hat{\pi}_{jt} : T \right\rangle \right| dt$$

$$\leqslant \sum_{j=1}^{k+1} \prod_{i \neq j} \mathrm{Lip}(\pi_i) \int_0^1 \int_X \left| f_t \frac{\partial \pi_{jt}}{\partial t} \right| d\|T\|\, dt$$

$$\leqslant (k+1) \prod_{j=1}^{k+1} \mathrm{Lip}(\pi_j) \int_0^1 \int_X |f|\, d\|T\|\, d\mathcal{L}^1$$

we find that $[0,1] \times T$ has finite mass and $\|[0,1] \times T\| \leqslant (\mathcal{L}^1 \llcorner [0,1]) \otimes \|T\|$. We now establish the equality

$$\langle f\, d\pi : \partial\big([0,1] \times T\big) + [0,1] \times \partial T \rangle = \langle f\, d\pi : 1 \times T - 0 \times T \rangle \qquad (*)$$

for the special case of f, π_j differentiable wrt. t, such that $\frac{\partial f_t}{\partial t}$, $\frac{\partial \pi_{jt}}{\partial t}$ are bounded Lipschitz functions. Indeed, $\langle f\, d\pi : t \times T \rangle$ is a Lipschitz function of t, and for a.e. $t \in [0,1]$,

$$\begin{aligned}
&\frac{\partial}{\partial t} \langle f\, d\pi : t \times T \rangle \\
&\quad = \lim_{h \to 0} \left\langle \frac{f_{t+h} - f_t}{h}\, d\pi_{t+h} : T \right\rangle + \\
&\qquad \sum_{j=1}^{k} (-1)^{j+1} \left\langle f_t\, d\left(\frac{\pi_{j,t+h} - \pi_{jt}}{h} \right) \wedge d\pi_{1t} \wedge \cdots d\pi_{j-1,t} \wedge d\pi_{j+1,t+h} \wedge \cdots d\pi_{k,t+h} : T \right\rangle \\
&\quad = \left\langle \frac{\partial f_t}{\partial t}\, d\pi_t : T \right\rangle + \sum_{j=1}^{k} (-1)^{j+1} \left\langle f_t\, d\left(\frac{\partial \pi_{jt}}{\partial t} \right) \wedge d\hat{\pi}_{jt} : T \right\rangle .
\end{aligned}$$

Thus, by the fundamental theorem of calculus,

$$\langle f\, d\pi : 1 \times T - 0 \times T \rangle = \int_0^1 \left[\left\langle \frac{\partial f_t}{\partial t}\, d\pi_t : T \right\rangle + \sum_{j=1}^{k} (-1)^{j+1} \left\langle f_t\, d\left(\frac{\partial \pi_{jt}}{\partial t} \right) \wedge d\hat{\pi}_{jt} : T \right\rangle \right] dt .$$

On the other hand, by the chain rule,

$$\begin{aligned}
&\langle f\, d\pi : \partial\big([0,1] \times T\big) + [0,1] \times \partial T \rangle \\
&\quad = \langle df \wedge d\pi : [0,1] \times T \rangle + \sum_{j=1}^{k} (-1)^{j+1} \int_0^1 \left\langle f_t \frac{\partial \pi_{jt}}{\partial t} : \partial T \right\rangle dt \\
&\quad = \int_0^1 \Bigg[\left\langle \frac{\partial f_t}{\partial t}\, d\pi_t : T \right\rangle + \\
&\qquad\qquad \sum_{j=1}^{k} (-1)^{j+1} \left(\left\langle d\left(f_t \frac{\partial \pi_{jt}}{\partial t} \right) \wedge d\hat{\pi}_{jt} : T \right\rangle - \left\langle \frac{\partial \pi_{jt}}{\partial t}\, df_t \wedge \hat{\pi}_{jt} : T \right\rangle \right) \Bigg] dt \\
&\quad = \langle f\, d\pi : 1 \times T - 0 \times T \rangle .
\end{aligned}$$

This proves $(*)$ in the special case.

We now prove $(*)$ in general and the continuity axiom for $[0,1] \times T$. Fix a Lipschitz function $\varrho : \mathbb{R} \to [0,1]$ with support in $[-1,1]$ such that $\varrho(0) = 1 = \int_0^2 \varrho \, d\mathcal{L}^1$. Note that for any bounded Lipschitz function $f : [0,1] \times X \to \mathbb{R}$,

$$f^\varepsilon(t,x) = \frac{1}{\varepsilon} \int_0^1 \varrho(\varepsilon(s-t)) f(s,x) \, ds \quad \text{for all } (t,x) \in [0,1] \times X, \, \varepsilon > 0,$$

defines a function f^ε differentiable w.r.t. t such that the t-derivative is bounded Lipschitz function. Moreover, $f^\varepsilon \to f$ $(\varepsilon \to 0+)$ with bounded Lipschitz constants.

Thus, if $[0,1] \times \partial T$ is a current, then by continuity, the equation holds for arbitrary $f \, d\pi$. But then $\partial([0,1] \times \partial T)$ is a current as the sum of currents. This implies the continuity axiom for $[0,1] \times \partial T$ in the case f a characteristic function. By approximation (the right hand side has finite mass because T is normal), it also holds in general. This shows by induction that it suffices to prove the statement for $k = 0$. But then $\partial T = 0$, and $[0,1] \times \partial T$ is a current.

Thus, T is a current, and we have the equation, for any k. But the equation shows that $\partial(T \times [0,1])$ has finite mass, since this is the case for ∂T. □

This theorem furnishes a surprisingly simple proof of the existence of filling currents, due to Stefan Wenger [Wen04]. We do not state it in its most general form.

Existence of Filling Currents 5.6.6. Let X be a Banach space and suppose that we are given $S \in \mathbf{I}_k(X)$, $\partial S = 0$, with $r = \operatorname{diam} \operatorname{supp} S < \infty$. Then there exists $T \in \mathbf{I}_{k+1}(X)$ such that we have $\operatorname{supp} T \subset \overline{\operatorname{co}(\operatorname{supp} S)}$, $\partial T = S$, and

$$\|T\| \leqslant (k+1) r \|S\|, \quad \text{in particular,} \quad \mathbf{M}(T) \leqslant (k+1) r \mathbf{M}(S).$$

Proof. Fix $x_0 \in \operatorname{supp} T$. Let $\varphi : [0,1] \times \operatorname{supp} T \to X$ be defined by

$$\varphi(t,x) = t(x - x_0) + x_0 \quad \text{for all } x \in \operatorname{supp} S.$$

Then $\operatorname{Lip}(\varphi(\sqcup, x)) \leqslant r$ for all $x \in \operatorname{supp} T$, and $\operatorname{Lip}(\varphi(t, \sqcup)) \leqslant t \leqslant 1$ for all $t \in [0,1]$. Let $T = \varphi_\bullet([0,1] \times S)$. Then $T \in \mathbf{I}_{k+1}(X)$ and

$$\partial T = \varphi_\bullet(\partial([0,1] \times S)) = \varphi_\bullet(1 \times S) - \varphi_\bullet(0 \times S)$$

by theorem 5.6.5. Moreover,

$$\langle f \, d\pi : \varphi_\bullet(t \times S) \rangle = \langle f \circ \varphi_t \, d(\pi \circ \varphi_t) : S \rangle,$$

so $\varphi_\bullet(1 \times S) = S$ ($\varphi_1 = \mathrm{id}$) and $\varphi_\bullet(0 \times S) = 0$ (φ_0 constant). Moreover,

$$|\langle f \, d\pi : T \rangle| \leqslant \sum_{j=1}^{k+1} \int_0^1 \left\langle f \circ \varphi_t \frac{\partial(\pi_j \circ \varphi_t)}{\partial t} \, d\widehat{\pi \circ \varphi_t}_j : T \right\rangle dt$$

$$\leqslant \sum_{j=1}^{k+1} \prod_{i\neq j} \mathrm{Lip}(\pi_i \circ \varphi_t) \int_0^1 \int_X \left| f \circ \varphi_t \frac{\partial(\pi_j \circ \varphi_t)}{\partial t} \right| d\|T\|\, dt$$

$$\leqslant (k+1) r \prod_{j=1}^{k+1} \mathrm{Lip}(\pi_j) \cdot \int_0^1 \int_X |f \circ \varphi|\, d\|T\|\, d\mathcal{L}^1 ,$$

so the assertion follows by definition of mass. □

Remark 5.6.7.

(i). In the proof, φ can be replaced by any Lipschitz contraction. Thus, the results to all metric spaces with a uniform bound on the Lipschitz constants of contractions of bounded subsets, cf. [Wen04, prop. 5.4]. (Such a metric space is, in particular, contractible.)

(ii). Note that the construction in the proof amounts to 'suspension' within the space of integral currents.

(iii). A more precise statement, the so-called isoperimetric inequality', would remove the dependence on r at the expense of including the power $\frac{k+1}{k}$ on the right hand mass. Such a theorem can be deduced from the above statement, and forms the main portion of Wenger's thesis [Wen04]. However, the above statement suffices for the solution of Plateau's problem in the case of Hilbert spaces.

Proof of theorem 5.6.2. Let $r = \operatorname{diam} \operatorname{supp} S$, which is finite. By theorem 5.6.6, the set

$$Y = \{ T \in \mathbf{I}_{k+1}(X) \mid \partial T = S\, , \mathbf{M}(T) \leqslant r(k+1)\mathbf{M}(S) \}$$

is non-empty. In particular, $M := \inf \mathbf{M}(Y) = \inf\{ \mathbf{M}(T) \mid T \in \mathbf{M}(X)\, , \partial T = S \}$. Let $T_j \in Y$, $\lim_j \mathbf{M}(T_j) = M$. Replacing T_j by $\pi_{K\bullet}(T_j)$ where $K = \mathrm{co}(\operatorname{supp} S)$ is compact and $\pi_K : X \to K$ is the metric projection, which is 1-Lipschitz, and characterised by

$$\|\pi_K(x) - x\| = \mathrm{dist}(K, x) \quad \text{for all } \ x \in X\, ,$$

we may assume $\operatorname{supp} T_j \subset K$. Then $\sup_j \|T_j\| \leqslant M + \mathbf{M}(S) < \infty$. By the compactness theorem 5.4.2, extracting a subsequence, we may assume $T_j \to T \in \mathbf{N}_{k+1}(X)$ pointwise, and $\operatorname{supp} T \subset K$ is compact. In particular, $\partial T = \lim_j \partial T_j = S$. By the closure theorem 5.5.10, $T \in \mathbf{I}_{k+1}(X)$. This proves the theorem. □

Remark 5.6.8. The solution of Plateau's problem in Hilbert spaces is from [AK00a], the modified cone construction and the existence of filling currents are from Wenger's thesis [Wen04].

5.7 Intrinsic Representation of Currents

5.7.1. Having reformulated and solved Plateau's problem in the framework of currents, it is important to ensure that the result is indeed what we intended it to be, namely, we need to see how integral currents correspond to countably k-rectifiable sets. We shall not address the question of regularity of minimal filling surfaces, which is rather delicate, albeit natural, given the classical formulation of Plateau's problem.

5.7.2. We assume now that X be embedded in some $Y = E^*$ where E is a separable Banach space. Let $\tau = \tau_1 \wedge \cdots \wedge \tau_k \in \bigwedge^k Y$ be a simple k-vector. Let J_τ be defined by $J_\tau = \mathrm{J}_k(L_\tau)$ where $L_\tau : \mathbb{R}^k \to Y$ is given by $L_\tau(x) = \sum_{j=1}^k x_j \tau_j$. Then J_τ is well-defined, since for $\sigma_i = \lambda_i \tau_i$, defining $M_\lambda(x) = (\lambda_i x_i)$, we have

$$J_{\tau_1 \wedge \cdots \wedge \tau_k} = \prod_{j=1}^k |\lambda_j| \cdot J_{\sigma_1 \wedge \cdots \wedge \sigma_k} \,.$$

The simple k-vector τ is said to a unit simple k-vector if $J_\tau = 1$. Clearly, any simple k-vector is equivalent to a normalised one.

If $S \subset X$ is countably k-rectifiable, then an *orientation* is a function

$$\tau = \tau_1 \wedge \cdots \wedge \tau_k : S \to \bigwedge^k Y$$

such that τ_j is a Borel function for each j , $\tau(x)$ is a unit simple k-vector and $\mathrm{Tan}^k(S, x)$ is the span of $\tau_j(x)$, $j = 1, \ldots, k$, for $\mathcal{H}^k$ a.e. $x \in S$.

Given a Lipschitz map $\pi : S \to \mathbb{R}^p$, we define for $\mathcal{H}^k$ a.e. $x \in S$ a p-covector $\wedge_p \, d^S\pi(x)$ by

$$\wedge_p \, d^S\pi(x) = d^S\pi_1(x) \wedge \cdots \wedge d^S\pi_p(x) \in \bigwedge^p \mathrm{Tan}^k(S, x)^* \,.$$

If $p = k$ and $f : \mathbb{R}^k \to X$ is a Lipschitz map, then for $\mathcal{H}^k$ a.e. $y = f(x) \in S$, the defining (chain) rule for the tangential differential gives

$$\mathrm{J}_k(d^S\pi(x)) = \frac{|\det(\pi \circ f)'(x)|}{\mathrm{J}_k(f'_\sigma(x))} = \left|\left\langle J_{\tau_x}^{-1}\tau_x : \wedge_p d^S\pi(y)\right\rangle\right|$$

where $\tau_x = f'_\sigma(x)e_1 \wedge \cdots \wedge f'_\sigma(x)e_k$ and the duality between k-vectors and k-covectors is given as usual by

$$\langle x_1 \wedge \cdots \wedge x_k : \xi^1 \wedge \cdots \wedge \xi^k \rangle = \det(\langle x_i : \xi^j \rangle) \quad \text{for all } \; x_i \in V \,, \, \xi^j \in V^*$$

where V is any vector space. Since f was arbitrary, we conclude that for any orientation σ of S ,

$$\mathrm{J}_k(d^S\pi(x)) = \left|\left\langle \sigma(x) : \wedge_k \, d^S\pi(x)\right\rangle\right| \quad \text{for } \mathcal{H}^k \text{ a.e.} \quad x \in S \,.$$

Proposition 5.7.3. Let $S \subset Y = E^*$ be countably k-rectifiable where E is a separable Banach space, $\vartheta : S \to]0,\infty[$ a $\mathcal{H}^k$ summable Borel function, and τ an orientation of S. Then

$$\langle f\, d\pi : [\![S,\vartheta,\tau]\!]\rangle = \int_S f(x)\vartheta(x)\langle \tau(x) : \wedge_k d^S\pi(x)\rangle\, d\mathcal{H}^k(x) \quad \text{for all } f\,d\pi \in \mathcal{D}^k(X)$$

determines $[\![S,\vartheta,\tau]\!] \in \mathcal{R}_k(Y)$. Moreover, this current is in $\mathcal{I}_k(Y)$ if ϑ maps to $\mathbb{N}\setminus 0$. Conversely, any $T \in \mathcal{R}_k(Y)$ (resp. $T \in \mathcal{I}_k(Y)$ is given in this way.

Proof. First, we prove that any $T \in \mathcal{R}_k(Y)$ has this form. To that end, we consider $T = \varphi_\bullet[\![g]\!]$ where $g \in \mathbf{L}^1(\mathbb{R}^k,\mathbb{R})$ (resp. $\mathbf{L}^1(\mathbb{R}^k,\mathbb{Z})$) is supported in some compact K, and $\varphi \in \operatorname{Lip}(\mathbb{R}^k,Y)$ is bi-Lipschitz on K. Let $L = \varphi(K)$ and τ any orientation of L. Let $\eta_x = \mathrm{J}_k(f'_\sigma(x))^{-1}\cdot \varphi'_\sigma(x)e_1\wedge\cdots\wedge\varphi'_\sigma(x)e_k$. Then $\eta_x = \sigma(x)\tau_{\varphi(x)}$ for some $\sigma(x) = \pm 1$ and

$$\det(\pi\circ\varphi)'(x) = \sigma(x)\cdot\langle \tau_{\varphi(x)} : \wedge_k d^S\pi(\varphi(x))\rangle\cdot \mathrm{J}_k(f'_\sigma(x)) \quad \text{for } \mathcal{H}^k \text{ a.e.} \quad x\in K\,.$$

Then

$$\begin{aligned}\langle f\,d\pi : T\rangle &= \int_K g\cdot(f\circ\varphi)\cdot\det(\pi\circ\varphi)'\,d\mathcal{L}^k\\ &= \int_L f(y)g(\varphi^{-1}(y))\sigma(\varphi^{-1}(y))\langle\tau_y : \wedge_k d^S\pi(y)\rangle\,d\mathcal{H}^k(y)\end{aligned}$$

by the change of variables formula corollary 4.1.6. Setting $\vartheta(y) = g(x)\sigma(x)$ for $y = \varphi(x)$ (possibly changing the sign of τ) and $S = L\cap\{\vartheta > 0\}$, the assertion follows in the special case. The general case follows by approximation with normal currents, i.e. theorem 5.3.5.

That any triple as above induces a rectifiable current (resp. an integer-rectifiable current) in the case $E = \mathbb{R}^k$ follows just as in proposition 5.2.3. The case $S = \varphi_\bullet(K)$ where K is some compact in $\mathbb{R}^k$ and φ is bi-Lipschitz then follows by the area formula. The general case follows, as usual, from lemma 4.2.3. □

5.7.4. To give a definitive statement of the above proposition in a general metric framework, we introduce the following concepts. A triple (S,ϑ,τ) where S is a countably k-rectifiable set in $Y = E^*$, $\vartheta : S \to]0,\infty[$ an $\mathcal{H}^k$ summable Borel function, and τ an orientation of S, is called an *oriented k-rectifiable set*. Two such oriented k-rectifiable sets (S_j,ϑ_j,τ_j), $j = 1,2$, are said to be *equivalent* if there exist subsets $S'_j \subset S_j$, $\mathcal{H}^k(S_j\setminus S'_j) = 0$, and an isometric bijection $f : S'_1 \to S_2$ satisfying $\vartheta_1 = \vartheta_2\circ f$ and

$$\wedge^k d^{S_1} f(x)\tau_1(x) = \tau_2(x) \quad \text{for all } x\in S_1\,,$$

$\wedge^k L : \bigwedge^k V \to \bigwedge^k W$ being the linear map induced by any linear $L : V\to W$.

Theorem 5.7.5. Let (S_j,ϑ_j,τ_j) be oriented k-rectifiable sets in dual Banach spaces $Y_j = E_j^*$. Then they are equivalent if and only if there is an isometry $i : S'_1 \to S'_2$, $\mathcal{H}^k(S_j\setminus S'_j) = 0$, such that $\vartheta_1 = \vartheta_2 \circ i$ and

$$i_\bullet [\![S_1, \vartheta_1, \tau_1]\!] = [\![S_2, \vartheta_2, \tau_2]\!] \,.$$

In particular, they are equivalent if they represent the pushforward $i_{j\bullet} T$ of the same rectifiable current $T \in \mathcal{R}_k(X)$, X some metric space.

5.7.6. We shall not prove the theorem. The proof is based on standard arguments.

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